

All necessary work must be shown for credit and must represent the question asked. Your work MUST be NEAT. You may NOT use computers, notes or texts. Calculators can be used only to help with arithmetic.

I have neither received nor given help on this exam. Don Key  
(Signature) (2 points)

1. Give the output of the following Matlab script. (8 points)

```
prod = 1;
for k = 1:2
    for j = 1:k-1
        prod = prod*j;
    end
    for j = k+1:3
        prod = prod*j;
    end
end
disp(prod);
```

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PROD	K	V	
1			
1	1	1; 0 *	
1.2	2	2:3	
1.2.3	3	1:1	
1.2.3.1	2	3:3	
1.2.3.1.3	1		
18	18		

2. Let  $p(x) = \sum_{k=0}^{10} (-1)^k k! x^{k+1}$ . Give  $p(0)$ ,  $p'(0)$ ,  $p^{(5)}(0)$ ,  $p^{(11)}(0)$ . (8 points)

$$= x - x^2 + 2x^3 - 3!x^4 + 4!x^5 + \dots + 10!x^{11}$$

$$p(0) = 0, \quad p'(0) = 1, \quad p^{(5)}(0) = 5! a_5 = 5! 4!, \quad p^{(11)}(0) = 11! a_{11} = 11! 10!$$

3. Consider the points  $\{(-1,1), (1,3)\}$ . Give (2 points each)

(a) the Newton interpolating polynomial

X	Y	$\Delta$
-1	1	
1	3	$\frac{3-1}{1-(-1)} = 1$

$$N_1(x) = 1 + 1(x - -1) = 1 + x + 1 \\ = x + 2$$

(b) the Lagrange interpolating polynomial

$$P_1(x) = 1 \frac{(x-1)}{(-1-1)} + 3 \frac{(x-1)}{(1-1)} = -\frac{1}{2}(x-1) + \frac{3}{2}(x+1)$$

(c) the Vandermonde interpolating polynomial.

$$V_1(x) = a_0 + a_1 x \quad V_1(1) = a_0 + a_1 = 3 \\ V_1(-1) = a_0 - a_1 = 1 \quad 2a_0 = 4, \quad a_0 = 2, \quad a_1 = 1$$

(d) SHOW all three are the same.

$$V_1(x) = 2 + x$$

$$N_1(x) = x + 2, \quad P_1(x) = -\frac{1}{2}x + \frac{1}{2} + \frac{3}{2}x + \frac{3}{2} = x + 2 = V_1(x)$$

4. Let  $f(x) = e^{-2x}$ . (24 points)

(a) Give the 8<sup>th</sup> degree Maclaurin polynomial for  $f$ .

$$f'(x) = -2e^{-2x}$$

$$a_0 = f(0) = 1$$

$$M_8(x) = 1 - 2x + \frac{(-2)^2}{2} x^2 + \frac{(-2)^3}{3!} x^3 +$$

$$f''(x) = (-2)^2 e^{-2x}$$

$$a_1 = f'(0) = -2$$

$$\dots + \frac{(-2)^8}{8!} x^8$$

$$f'''(x) = (-2)^3 e^{-2x}$$

$$a_2 = \frac{f''(0)}{2} = \frac{(-2)^2}{2}$$

$$= \sum_{k=0}^8 \frac{(-2)^k}{k!} x^k$$

$$f^{(k)}(x) = (-2)^k e^{-2x}$$

$$a_3 = \frac{f'''(0)}{3!} = \frac{(-2)^3}{3!}$$

$$a_k = \frac{f^{(k)}(0)}{k!} = \frac{(-2)^k}{k!}$$

(b) Use this polynomial to give an approximation for  $e$ .

$$\begin{aligned} f\left(\frac{-1}{2}\right) &= e^{-2\left(\frac{-1}{2}\right)} = e \approx M_8\left(\frac{-1}{2}\right) \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} \end{aligned}$$

(c) Give an upper error bound for this approximation.

$$\text{Error} \leq \frac{1}{(n+1)!} |f^{(n+1)}(c)| |x|^{n+1} \quad n = 8$$

$$\leq \frac{1}{9!} \left| (-2)^9 e^{-2c} \left| \frac{-1}{2} \right|^9 \right| \leq \frac{e^{-2c}}{9!} \leq \frac{e}{9!}$$

(d) SHOW what the interval of convergence is for the Maclaurin (power) series of  $f$ .

$$a_n = \frac{(-2)^n}{n!} \quad a_{n+1} = \frac{(-2)^{n+1}}{(n+1)!}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| / |x| = \lim_{n \rightarrow \infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{n!}{2^n}} |x|$$

$$= |x| \lim_{n \rightarrow \infty} \frac{2}{n+1} = 0 \quad \text{for all } x$$

$$-\infty < x < \infty$$

5. Give the second order Taylor polynomial approximation for the fixed point function in Newton's method for determining the  $\sqrt{2}$ . (12 points)

$$x = \sqrt{2}$$

$$x^2 - 2 = 0$$

$$g(x) = x^2 - 2$$

$$g'(x) = 2x$$

$$f(x) = x - \frac{g(x)}{g'(x)} = x - \frac{x^2 - 2}{2x} = x - \left(\frac{1}{2}x - \frac{1}{x}\right) \\ = \frac{1}{2}x + \frac{1}{x}$$

$$f'(x) = \frac{1}{2} - \frac{1}{x^2}$$

$$f''(x) = \frac{2}{x^3}$$

$$T_2(x, \sqrt{2}) = f(\sqrt{2}) + f'(\sqrt{2})(x - \sqrt{2}) + \frac{1}{2}f''(\sqrt{2})(x - \sqrt{2})^2 = \sqrt{2} +$$

6. Give a 4<sup>th</sup> degree Taylor polynomial that can be used to determine an accurate approximation to  $\sqrt{2}$ . What is the approximation? (12 points)

$$f(x) = \sqrt{x} \quad x_0 = 1 \text{ is close to 2}$$

$$a_0 = f(1) = 1$$

$$a_1 = f'(1) = \frac{1}{2}$$

$$a_2 = \frac{f''(1)}{2} = \frac{-1}{2^3}$$

$$a_3 = \frac{f'''(1)}{3!} = \frac{3}{2^3 3!} = \frac{1}{2^4}$$

$$a_4 = \frac{f^{(4)}(1)}{4!} = -\frac{15}{4! 2^4}$$

$$T_4(x; 1) = 1 + \frac{1}{2}(x-1) - \frac{1}{2^3}(x-1)^2 + \frac{1}{2^4}(x-1)^3 - \frac{15}{4! 2^4}(x-1)^4$$

$$f(2) = \sqrt{2} \approx T_4(2; 1) = 1 + \frac{1}{2} - \frac{1}{2^3} + \frac{1}{2^4} - \frac{15}{4! 2^4} \approx 1.398$$

7. Consider the points  $S = \{(-1, 2), (0, 5), (1, 0), (2, -1)\}$ . Give (8 points each) (You don't have to multiply out.)  
 (a) the Newton interpolating polynomial and its derivative at  $x = 0$  or an approximation to this derivative value.

$x$	$y$	$\Delta$	$\Delta^{(2)}$	$\Delta^{(3)}$			
-1	2						
0	5	3					
1	0	-5	-4				
2	-1	-1	2	2			

$$N_3(x) = 2 + 3(x+1) - 4(x+1)(x+0) + 2(x+1)(x+0)(x-1)$$

$$N_3'(0) \approx \Delta(0) = -5$$

$$N_3'(x) = 3 - 4(x+1) + 2(x+1)(x-1)$$

$$N_3'(0) = 3 - 4 + 2(-1) = -3$$

(b) the Lagrange interpolating polynomial

$$2 \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} + 5 \frac{(x-1)(x-0)(x-2)}{(0-1)(0-0)(0-2)} + \\ 0 + (-1) \frac{(x-1)(x-0)(x-1)}{(2-1)(2-0)(2-1)}$$

(c) the Vandermonde matrix and vector for the Vandermonde interpolating polynomial

$$V_3 = \begin{pmatrix} 1 & -1 & (-1)^2 & (-1)^3 \\ 1 & 0 & 0^2 & 0^3 \\ 1 & 1 & 1^2 & 1^3 \\ 1 & 2 & 2^2 & 2^3 \end{pmatrix} \quad \bar{Y} = \begin{pmatrix} 2 \\ 5 \\ 0 \\ -1 \end{pmatrix}$$

BP: (Two Points) Give the 15<sup>th</sup> degree Maclaurin polynomial for  $f(x) = \frac{1}{1+x}$ .

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - x^7 + x^8 - x^9 + x^{10} - x^{11} \\ + x^{12} - x^{13} + x^{14} - x^{15}$$