MATH 248 SPRING 2017 – LABORATORY ASSIGNMENT 6 – Sochacki
DUE: Tuesday November 28, 2017
POINTS: 50

You are to write a Matlab script that will determine approximations to the first and second derivatives for the solution to the initial value ordinary differential equation (IVODE) problem

$$x'(t) = a * x(t) - b;\ x(0) = \alpha$$

at a given t. You should ask the user for the numbers a, b, α, t and other parameters needed to approximate the first and second derivative.

Guidelines:

(1) First you should do a neat one-three page (8.5 x 11) write up showing properties of the solution to this system. You should sketch what solutions to this equation look like. These sketches should be accurate. Since you can solve this equation exactly, you should give the error bounds for your approximations to the first and second derivatives of the solution.
(2) Your program should print the approximations to the derivatives of the solution with labels indicating what the output represents.
(3) You should make sure your code minimizes calculation and errors.
(4) Since you know the exact answer, you should have your code output the absolute error for your approximations.
(5) BP: Give a plot of your numerical solution to this IVODE for the parameters I give you. (2 points)

Your matlab codes should have variable names that are descriptive. Your coding should be top down and efficient. Make sure the number of calculations is minimized. Your input and output should be well labeled with easy to read instructions. Your turn in should be neat and professional.

LAB 6

$$X'(t) = a\,x(t) - b; \qquad X(0) = \alpha$$

$$X''(t) = a\,X'(t)$$

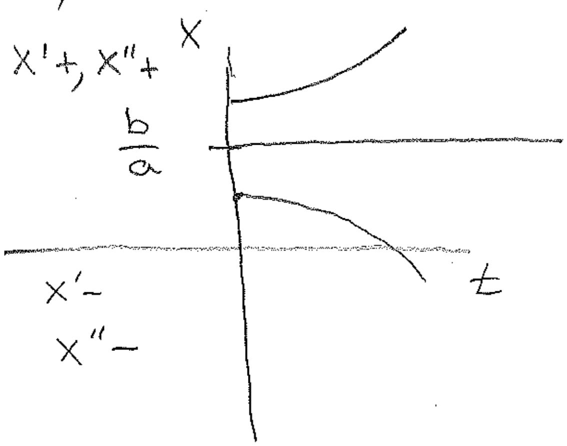
$$X'''(t) = a\,X''(t)$$

$$X^{(4)}(t) = a\,X'''(t)$$

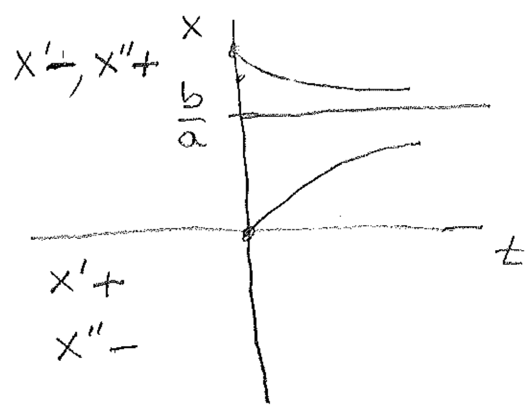
$$X'(t) = 0 = a\,x(t) - b \Rightarrow X(t) = \frac{b}{a}$$

$$X''(t) = 0 = a\,X'(t) \Rightarrow X'(t) = 0 \Rightarrow X(t) = \frac{b}{a}$$

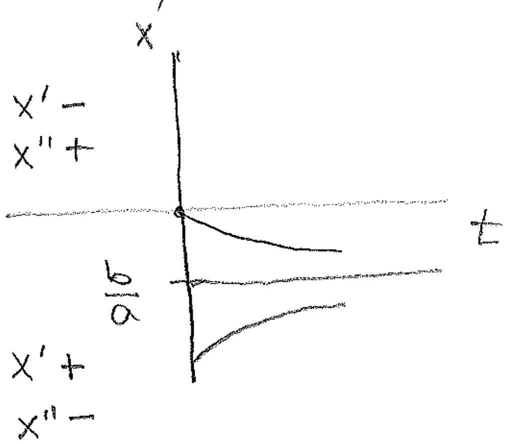
I) $a, b > 0$



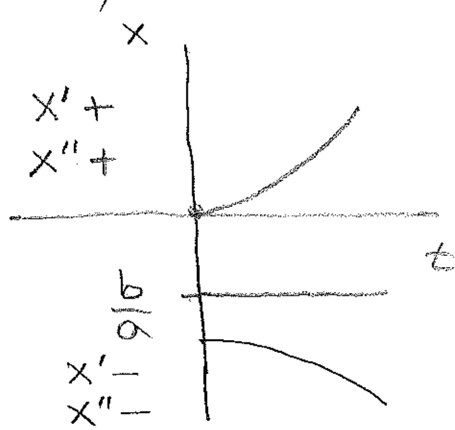
II) $a, b < 0$



III) $a < 0,\ b > 0$



IV) $a > 0,\ b < 0$



$$x'(t) = ax(t) - b$$

$$\frac{x'(t)}{ax(t)-b} = 1$$

$$\frac{a\,x'(t)}{ax(t)-b} = a$$

$$\int_0^t \frac{ax'(t)}{ax(t)-b}\,dt = \int_0^t a\,dt$$

$$\ln|ax(t)-b|\Big|_0^t = at\Big|_0^t$$

$$\ln|ax(t)-b| - \ln|ax(0)-b| = at$$

$$\ln|ax(t)-b| - \ln|a\alpha-b| = at$$

$$\ln\left|\frac{ax(t)-b}{a\alpha-b}\right| = at$$

$$\frac{ax(t)-b}{a\alpha-b} = e^{at}$$

$$ax(t)-b = (a\alpha-b)e^{at}$$

$$x(t) = \frac{(a\alpha-b)e^{at}+b}{a}$$

$$x'(t) = (a\alpha - b)e^{at}$$

$$x''(t) = a(a\alpha - b)e^{at}$$

$$x'''(t) = a^2(a\alpha - b)e^{at}$$

$$x^{(4)}(t) = a^3(a\alpha - b)e^{at}$$

<u>FD</u>

$$x'(t) \approx \frac{x(t+h) - x(t)}{h} \approx ax(t) - b$$

$$x(t+h) \approx x(t) + h.(a.x(t) - b) \quad (?)$$

$$= (1+ah).x(t) - h.b$$

OR

$$\frac{x(t+h) - x(t)}{h} = \frac{\frac{(a\alpha-b)e^{a(t+h)}+b}{a} - \frac{(a\alpha-b)e^{at}+b}{a}}{h}$$

$$\text{ERROR} \leq \frac{1}{2} M_2 h$$

$$\leq \frac{1}{2} a|(a\alpha - b)|e^{at_*} h$$

BD

$$X'(t) \approx \frac{X(t)-X(t-h)}{h} \approx aX(t)-b$$

$$X(t) \approx X(t-h)+h\big(aX(t)-b\big)$$

$$X(t)(1-ah) \approx X(t-h)-hb$$

$$X(t) \approx \frac{X(t-h)-hb}{1-ah}$$

OR

$$\frac{X(t)-X(t-h)}{h} = \frac{\dfrac{(a\alpha-b)e^{at}+b}{a} - \dfrac{(a\alpha-b)e^{a(t-h)}+b}{a}}{h}$$

$$\text{ERROR} \le \tfrac{1}{2} M_2 h$$

$$\le \tfrac{1}{2}\, a\big|(a\alpha-b)\big| e^{at_*}\, h$$

CD

$$x'(t) \approx \frac{x(t+h)-x(t-h)}{2h} \approx ax(t)-b$$

$$x(t+h) \approx x(t-h) + 2h(ax(t)-b)$$

OR

$$\frac{x(t+h)-x(t-h)}{h} = \frac{\frac{(a\alpha-b)e^{a(t+h)}+b}{a} - \frac{(a\alpha-b)e^{a(t-h)}+b}{a}}{h}$$

$$\text{Error} \leq \frac{1}{3!} M_3 h^2$$

$$\leq \frac{1}{6} a^2 |(a\alpha-b)| e^{at_*} h^2$$

CD2

$$x''(t) \approx \frac{x(t+h)-2x(t)+x(t-h)}{h^2}$$

$$= \left[\frac{(a\alpha-b)e^{a(t+h)}+b}{a} - 2\,\frac{(a\alpha-b)e^{at}+b}{a} + \frac{(a\alpha-b)e^{a(t-h)}+b}{a}\right]\frac{1}{h^2}$$

$$\text{Error} \le \frac{1}{4!} M_4 h^2$$

$$\le \frac{1}{24} \left| a^3 (a\alpha - b) \right| e^{a t_*} h^2$$

$x' = ax - b \qquad x(0) = \alpha$

$$x(t) = \frac{(a\alpha - b)e^{at} + b}{a}$$

1) $a = -2, \; b = -4, \; x(0) = 4, \; h = 0.125, \; Tf = 4$

$t = 0 : 0.125 : 4 \qquad \text{length}(t) = \boxed{33}$

$x'(t) = -2x(t) + 4, \; x'(t) = 0 \Rightarrow x(t) = -2$

$$x(t) = \frac{(-2(4)+4)e^{-2t} - 4}{-2} = 2e^{-2t} + 2$$

$x \to 2 \quad \text{as} \quad t \to \infty$

2) $a = -2, \; b = 4, \; x(0) = -2, \; h = 0.25, \; Tf = 4$

$t = 0 : 0.25 : 4 \qquad \text{length}(t) = \boxed{17}$

$x'(t) = -2x(t) - 4, \; x'(t) = 0 \Rightarrow x(t) = -2$

$x(t) = -2$

3) $a = 4, \; b = -4, \; x(0) = 0, \; h = 0.25, \; Tf = 8$

$t = 0 : 0.25 : 8 \qquad \text{length}(t) = \boxed{33}$

$x'(t) = 4x(t) + 4; \qquad x'(t) = 0 \Rightarrow x(t) = -1$

$$x(t) = \frac{4e^{4t} - 4}{4} = e^{4t} - 1$$

4) $a = 4, \; b = -4, \; x(0) = -2, \; h = 0.125, \; T_f = 8$

$\text{length}(t) = \boxed{65}, \quad x'(t) = 4x(t) + 4$

$$x(t) = \frac{-4e^{4t} - 4}{4} = -e^{4t} - 1$$