The Quadratic Ordinary Differential Equation

The Fundamental Ordinary Differential Equation

All college students should know about exponential growth. This comes from understanding the ordinary differential equation (ODE)

\[ x'(t) = bx(t) + c \]

where \( x \) is a function of \( t \), \( x' \) denotes the derivative of \( x \) and \( b, c \) are real numbers. We will also use

\[ \frac{d}{dt}x \]

to denote the derivative of \( x \).

Students should also understand the initial condition

\[ x(0) = x_0 \]

for this ODE. Here \( x_0 \) is a real number. If \( b = 1 \) and \( c = 0 \) we have the famous ODE

\[ x' = x; \ x(0) = x_0. \]

The solution to this ODE is

\[ x = e^t x_0, \]

since, if \( x = e^t x_0 \) then \( x' = e^t x_0 = x \) and \( x(0) = e^0 x_0 = x_0. \)

This ODE is important since the solutions grow toward \( \infty \) or \( -\infty \) as \( t \to \infty \) depending on the sign of \( x_0 \). Many things in our daily lives follow this exponential growth and are the source of many of our most serious and devastating problems. If \( b = -1 \) and \( c = 0 \) we have the famous ODE

\[ x' = -x; \ x(0) = x_0. \]

The solution to this ODE is

\[ x = e^{-t} x_0, \]

since, if \( x = e^{-t} x_0 \) then \( x' = -e^{-t} x_0 = -x \) and \( x(0) = e^0 x_0 = x_0. \)

This ODE is important since the solutions tend toward 0 independent of the sign of \( x_0 \) as \( t \to \infty \). Again, many things in our daily lives follow this exponential decay and are the source of many of our most serious and devastating problems.

In calculus, students learn that setting the derivative to zero gives much information about a function. In the ODE
\[ x' = bx + c \]

one has \( x' = 0 \) when \( x(t) = -\frac{c}{b} \). This constant solution is also known as an equilibrium solution. Another solution to this ODE is

\[ x = e^{bt}(x_0 + \frac{c}{b}) - \frac{c}{b}. \]

This solution can be obtained by noting that

\[ x'(t) = bx(t) + c \]

is equivalent to

\[ \frac{x'(\tau)}{bx(\tau) + c} = 1 \]

and that the Fundamental Theorem of Integral Calculus gives

\[
\int_0^t \frac{x'(\tau)}{bx(\tau) + c} \, d\tau = \int_0^t 1 \, d\tau \\
\frac{1}{b} \ln |bx(\tau) + c|_0^t = \tau|_0^t \\
\ln |bx(t) + c| - \ln |c| = bt.
\]

Solving this last equation for \( x \) gives us the general solution above.

It can be shown that this is the only form for solutions to this ODE. We now note that if \( b > 0 \) then the solutions will grow toward \( \pm \infty \) and if \( b < 0 \) then the solutions tend toward the equilibrium solution \( -\frac{c}{b} \). If \( b = 0 \), the ODE is \( x' = c \) and the solution is \( x = ct + x_0 \) which approaches \( \pm \infty \) as \( t \to \infty \).

We now consider polynomial solutions to this ODE. We suppose that

\[ x = x_0 + x_1 t + x_2 t^2 + ... = \sum_{k=0} x_k t^k \]

is a solution to the ODE. For this form of the solution, we have

\[ x' = x_1 + 2x_2 t + 3x_3 t^2 + ... = \sum_{k=0} (k + 1)x_{k+1} t^k. \]

Substituting these into the ODE \( x' = x \) gives

\[ x' = x_1 + 2x_2 t + 3x_3 t^2 + ... = x_0 + x_1 t + x_2 t^2 + ... = x \]

Equating the coefficients of like powers of \( t \) leads to
\[
x_1 = x_0 \\
x_2 = \frac{x_1}{2} = \frac{x_0}{2} \\
x_3 = \frac{x_2}{3} = \frac{x_0}{3!} \\
x_4 = \frac{x_3}{4} = \frac{x_0}{4!} \\
x_k = \frac{x_0}{k!}
\]

Substituting these coefficients back into the polynomial form for \( x \) gives

\[
x = x_0 + x_1 t + x_2 t^2 + \ldots = x_0 + x_0 t + \frac{x_0}{2} t^2 + \ldots + \frac{x_0}{k!} t^k = x_0 \sum_{k=0}^{\infty} \frac{1}{k!} t^k = x_0 \sum_{k=0}^{\infty} \frac{1}{k!} t^k.
\]

The last equality is true because \( x_0 = x_0 \). This polynomial is just the Maclaurin polynomial for \( xe^t \) which was shown above to be the solution to this ODE.

If we substitute the polynomial forms for \( x \) and \( x' \) into the general ODE \( x' = bx + c \) we determine that

\[x' = x_1 + 2x_2 t + 3x_3 t^2 + \ldots = b \left( x_0 + x_1 t + x_2 t^2 + \ldots \right) + c = bx + c.\]

Collecting the coefficients of like powers of \( t \) in this equation and equating as we did above gives us

\[
x_1 = bx_0 + c \\
x_2 = b \frac{x_1}{2} = b \frac{bx_0 + c}{2} \\
x_3 = b \frac{x_2}{3} = b^2 \frac{bx_0 + c}{3!} \\
x_4 = b \frac{x_3}{4} = b^3 \frac{bx_0 + c}{4!} \\
x_k = b^{k-1} \frac{bx_0 + c}{k!}
\]

Substituting these coefficients back into the polynomial form for \( x \) gives

\[
x = x_0 + (bx_0 + c)t + b \frac{bx_0 + c}{2} t^2 + \ldots = x_0 + \sum_{k=1}^{\infty} b^{k-1} \frac{bx_0 + c}{k!} t^k,
\]

for the general Maclaurin polynomial form of the solution \( x \) to the ODE \( x' = cx + b \) with initial condition \( x(0) = x_0 \). We note that the Maclaurin power series defined by this Maclaurin polynomial converges for all \( t \).
It will be shown that Maclaurin polynomial solutions can be obtained for many ODEs through the remaining ODEs presented in this work. From these ODEs one will see that generating Maclaurin polynomial solutions to ODEs is a natural mathematical process.

The Quadratic ODE

In fact, we apply what we have done above to obtaining the power series for the tangent function. We first note that if \( h(t) = f(t)/g(t) \) and

\[
\begin{align*}
  f(t) &= \sum_{i=0}^\infty f_i t^i; \quad g(t) = \sum_{i=0}^\infty g_i t^i; \quad h(t) = \sum_{i=0}^\infty h_i t^i
\end{align*}
\]

then \( f(t) = g(t)h(t) \) and

\[
\begin{align*}
  f(t) &= \sum_{i=0}^\infty f_i t^i = g(t)h(t) = \sum_{i=0}^\infty g_i t^i \sum_{i=0}^\infty h_i t^i.
\end{align*}
\]

Cauchy’s formula for the product of power series leads to

\[
\sum_{i=0}^\infty g_i t^i \sum_{i=0}^\infty h_i t^i = (g_0 + g_1 t + g_2 t^2 + \ldots)(h_0 + h_1 t + h_2 t^2 + \ldots)
= g_0 h_0 + (g_0 h_1 + g_1 h_0)t + (g_0 h_2 + g_1 h_1 + g_2 h_0)t^2 + \ldots
= \sum_{i=0}^\infty \left[ \sum_{j=0}^i g_j h_{i-j} \right] t^i
= g_0 h_0 + \sum_{i=1}^\infty \left[ \sum_{j=0}^i g_j h_{i-j} \right] t^i.
\]

Equating the coefficients of equal powers of \( t \) for \( f(t) \) with \( g(t)h(t) \) and solving for the coefficients of \( h \) gives

\[
\begin{align*}
  h_0 &= \frac{f_0}{g_0},
  h_i &= \frac{1}{g_0} \left( f_i - \sum_{j=0}^{i-1} g_{j+1} h_{i-1-j} \right); \quad i = 1, 2, 3, \ldots
\end{align*}
\]

Since \( x = \frac{\sin t}{\cos t} \), we can obtain the power series for \( \tan t \) using the above recurrence relation with the power series for \( f(t) = \sin t \) and \( g(t) = \cos t \). We have

\[
\begin{align*}
  f(t) &= \sin t = t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 - \frac{1}{7!} t^7 + \ldots
  g(t) &= \cos t = 1 + \frac{1}{2} t^2 - \frac{1}{4!} t^4 + \ldots,
\end{align*}
\]

giving \( f_{2i} = 0, \quad f_{2i+1} = (-1)^i \frac{1}{(2i+1)!}, \quad g_{2i} = (-1)^i \frac{1}{(2i)!}, \quad \text{and} \quad g_{2i+1} = 0 \) for \( i = 0, 1, 2, 3, \ldots \).
Using this leads to

\[
\sum_{i=0}^{\infty} h_i t^i = \frac{\sum_{i=0}^\infty \frac{t^{2i+1}}{(2i+1)!}}{\sum_{i=0}^\infty \frac{t^{2i}}{(2i)!}}
\]

\[
h_0 = \frac{f_0}{g_0} = \frac{\sin(0)}{\cos(0)} = 0,
\]

\[
h_{2i} = \frac{1}{g_0} (f_{2i} - \sum_{j=0}^{2i-1} g_j h_{2i-1-j}) = 0 ; \ i = 1, 2, 3, ...
\]

\[
h_{2i-1} = \frac{1}{g_0} (f_{2i-1} - \sum_{j=0}^{2i-2} g_j h_{2i-2-j}) = (-1)^{i-1} \frac{1}{(2i - 1)!} \sum_{j=0}^{2i-2} g_j h_{2i-2-j} ; \ i = 1, 2, 3, ...
\]

Since \(\tan t\) is an odd function, \(h_{2i} = 0\) for all \(i\). We find for the first few odd indices of \(h\) using the above recurrence relation that \(h_1 = 1, h_2 = \frac{2}{15}, h_3 = \frac{17}{315}\). As many coefficients as are desired for \(\tan t\) can be determined by continuing this recurrence relation.

Another way one can determine a power series for \(x = \frac{f(t)}{g(t)}\), is to differentiate \(x\) and use \(w = \frac{1}{g(t)}\). That is,

\[
x' = \frac{f'g - fg'}{g^2} = (f'g - fg')w^2
\]

\[
w' = -w^2 g'.
\]

Notice that because \(f = \sin t\) and \(g = \cos t\) the above leads to

\[
x' = (f^2 + g^2)w^2 = w^2
\]

\[
w' = w^2 f
\]

\[
f' = g
\]

\[
g' = -f.
\]

If we let \(z = wf\) in these equations, we obtain

\[
z' = w'f + wf' = w^2 f^2 + wg = z^2 + 1.
\]

The last equality arises from the fact that \(z = wf\) and \(w = \frac{1}{g}\).

Since \(x = \tan t\) satisfies the IV ODE

\[
x' = 1 + x^2 ; \ x(0) = 0,
\]
one can substitute the power series $x = \sum x_i t^i$ into this ODE (with polynomial right hand side) to obtain the power series for $\tan t$. Doing this gives

$$x' = \sum_{i=0}^{\infty} (i+1)x_{i+1} t^i = 1 + x^2 = 1 + \left( \sum_{i=0}^{\infty} x_i t^i \right)^2 = 1 + x_0^2 + \sum_{i=1}^{\infty} \left[ \sum_{j=0}^{i} x_j x_{i-j} \right] t^i.$$

We obtained the last equality by using Cauchy products for power series. Equating the coefficients of like powers of $t$ on both sides of the equation leads to $x_1 = 1 + x_0^2$ and

$$x_{i+1} = \frac{1}{i+1} \sum_{j=0}^{i} x_j x_{i-j} \text{ for } i = 1, 2, 3...$$

Since $x_0 = \tan 0 = 0$, we have $x_1 = 1$ and using the above recurrence relation gives us all the other coefficients in the power series for $\tan t$.

We now extend this method of using Cauchy products of power series to determine power series solutions for ODE with quadratic polynomial right hand side, $ax^2 + bx + c$. That is, consider the IV ODE

$$x' = ax^2 + bx + c; \ x(0) = x_0.$$ (If $a = 1, b = 0, c = 1, x_0 = 0$, we have the ODE for $x = \tan t$.)

We can determine the power series for this general quadratic ODE in a straightforward manner. We substitute in $x = \sum x_i t^i$ and and use Cauchy products for power series to obtain

$$x' = \sum_{i=0}^{\infty} (i+1)x_{i+1} t^i = ax^2 + bx + c$$

$$= a(\sum_{i=0}^{\infty} x_i t^i)^2 + b \sum_{i=0}^{\infty} x_i t^i + c$$

$$= ax_0^2 + bx_0 + c + a \sum_{i=1}^{\infty} \sum_{j=0}^{i} x_j x_{i-j} t^i + b \sum_{i=1}^{\infty} x_i t^i$$

The recurrence relation obtained from this by equating coefficients of like powers of $t$ from $x'$ with the last equation above is

$$x_1 = ax_0^2 + bx_0 + c$$

$$x_{i+1} = \frac{1}{i+1} (a \sum_{j=0}^{i} x_j x_{i-j} + bx_i); \ i = 1, 2, 3...$$
This recurrence relation gives the coefficients for the power series of \(x\). We point out that a simple recurrence like the above is straightforward to obtain using Cauchy products for power series for all quadratic polynomials systems of IV ODEs. We will demonstrate this later in the manuscript.

The above recurrence relation gives that

\[
x = x_0 + (ax_0^2 + bx_0 + c)t + \left( a^2 x_0^3 + a \left( cx_0 + \frac{3}{2} b x_0^2 \right) + \frac{1}{2} b^2 x_0 + \frac{1}{2} bc \right) t^2 + \left( a^3 x_0^4 + a^2 \left( \frac{4}{3} cx_0^2 + 2b x_0^3 \right) + \left( \frac{1}{3} c^2 + \frac{4}{3} b c x_0 + \frac{7}{6} b^2 x_0^2 \right) a + \frac{1}{6} b^3 x_0 + \frac{1}{6} b^2 c \right) t^3 + \ldots
\]

In the last section we studied the ODE \(x' = bx + c\) and considered the power series solution for this ODE. We can see from the power series solution above the impact of \(ax^2\) has on the difference in the solutions. If we set \(a = 0\) in the above power series we obtain the power series in the last section. Even if \(a\) and \(x_0\) are close to 0 it can be seen that as \(t\) gets close to 1 and then greater than 1 the above power series solution diverges from the power series solution given in the last section for \(x' = bx + c; \ x(0) = x_0\). Having the power series solution allows one to analyze the impact of coefficients (parameters) in the ODE even for non-polynomial ODEs as will be shown in later sections.

Since the quadratic equation \(ax^2 + bx + c = 0\) has zero, one or two real roots, there are 3 basic types of solutions for the ODE \(x' = ax^2 + bx + c\). For the case of no real roots the parabola \(y = ax^2 + bx + c\) lies completely above the \(x\)-axis or completely below the \(x\)-axis. In the first case the solution to this ODE is always increasing and in the second case the solution is always decreasing. For the case of one real root the solution to this ODE is again always increasing or always decreasing since the graph of \(y\) lies above or below the \(x\)-axis except at the vertex of the parabola which lies on the \(x\)-axis. This one equilibrium solution is unstable. For the the case of two real roots one equilibrium solution is stable and the other is unstable. We present the solution of the ODE for each of these three cases.

Case 1. No roots: \(x' = ax^2 + bx + c = a((x - r_1)^2 + r_2^2)\)

Since \(ax^2 + bx + c\) has no real roots, it can be written as the sum of two squares with real numbers \(r_1\) and \(r_2\) (which compose the real and imaginary part of the complex roots, \(r_1 \pm i r_2\) for \(ax^2 + bx + c = 0\)) as shown in the last equality for the ODE above. When the ODE is expressed in this form it can be integrated to give

\[
x = r_1 + \tan a r_2^2 (t + C).
\]

Case 2. One root: \(x' = ax^2 + bx + c = a(x - r)^2\)

Since \(ax^2 + bx + c\) has only one real root, \(r\) it can be written as a perfect square. The solution for this ODE is
\[ x = \frac{ar t + rC - 1}{at + C}. \]

Case 3. \( x' = ax^2 + bx + c = a(x - r_1)(x - r_2) \)

Since \( ax^2 + bx + c \) has two real roots, \( r_1, r_2 \) it can be factored into these roots. The solution for this ODE is

\[ x = \frac{r_2 e^{aC(r_1 - r_2)} t + C(r_1 - r_2) - r_1}{e^{aC(r_1 - r_2)} t + C(r_1 - r_2) - 1}. \]

where \( C \) is determined by the initial condition \( x(0) = x_0 \) in all three cases.

We note that all solutions ‘blow up in finite \( t \)’. That is, there is a real number \( t_* \) so that \( \lim_{t \to t_*} x = \pm \infty \). Therefore, the power series given above for all the solutions for all three cases have a finite radius of convergence.

If \( a = 0, b \neq 0 \), we have the ODE we studied in the first example. It is seen in these first two examples that the properties of the solutions to the ODE change drastically as the right hand side of the solution is changed from a line \( (bx + c) \) to a quadratic polynomial \( (parabola \ ax^2 + bx + c) \) no matter the magnitude of \( a \). In the next example we will demonstrate that solutions are drastically different from those ODEs whose right hand side is slightly different from a line. At the end of this section we will show why this is the case.

We now consider a famous ODE from population dynamics. If \( x(t) \) represents the population of a species that competes among its own members then it reasonable to assume that \( x \) satisfies the IV ODE

\[ x' = \beta x - \alpha x^2 = x(\beta - \alpha x); \ x(0) = x_0. \]

The equilibrium solutions for this ODE are \( x = 0 \) and \( x = \frac{\beta}{\alpha} \). From the discussion above, one equilibrium is unstable and the other one is stable depending on \( a, b \). This makes sense in population dynamics because the population could die out or it could approach a livable equilibrium. (If \( x_0 = 0 \) there is nothing to study since the population will always be 0 in this case.) For example, if \( a, b \) are both positive then \( x = 0 \) is an unstable equilibrium and \( x = \frac{\beta}{\alpha} \) is a stable equilibrium. Therefore, in this case the population of the species will tend toward \( \frac{\beta}{\alpha} \).

We now look at the specific case where \( a = 256, b = 192 \). We assume that \( x = 1 \) represents one thousand species (\( x = 2 \) represents two thousand species) and so on. The stable equilibrium in this case is \( x = \frac{\beta}{\alpha} = \frac{3}{4} \) representing a population of 750. The IV ODE is

\[ x' = 192x - 256x^2 = x(192 - 256x); \ x(0) = x_0 = 0.125. \]

That is, we are assuming that there are initially 125 species living under this population process. Let us assume that \( w = x_0 + x_1 t \) is the solution to this problem. Then
\[ w' = x_1 = 192w - 256w^2 = 192(x_0 + x_1t) - 256(x_0 + x_1t)^2. \]

Collecting like powers of \( t \) in this equation leads to

\[
x_1 = 192x_0 - 256x_0^2 + (192x_1 - 256(2x_0x_1))t - 256x_1^2t^2 = x'(0) + At + Bt^2.
\]

Therefore, we should set \( x_1 = x'(0) = 192x_0 - 256x_0^2 \). However, doing this ignores the \( A \) and \( B \) terms arising from the ODE. This will lead to an error in using \( w = x_0 + x_1t = 0.125 + t \) as the solution. Therefore, we call this \( w \) an approximation to the solution \( x \).

We now use this \( w \) to generate an approximation to the solution \( x \) at various \( t \). In particular, we will do this at the \( t \) in the set \( \{ t_k | t_k = kh, \ k \in \mathbb{N} \} \) (here \( \mathbb{N} \) represents the set of natural numbers 1,2,3,...).

Here is the algorithm to obtain these approximations.

1. Let \( w_0 = x_0 = 0.125 \) and \( k = 1 \)
2. Let \( w_1 = x_1 = 192x_0 - 256x_0^2 \)
3. Let \( w(t) = x_0 + x_1t \).
4. Let \( w_k = w(h) \) and then increment \( k \) by 1.
5. Let \( x_0 = w_k \) and go to step (1) until \( k = K \).

This algorithm is known as the forward Euler approximation to the solution of the IV ODE. It is a first degree polynomial (or linear) approximation to the solution to the IV ODE. The updating of \( x_0 \) in the algorithm above (and all the ones presented in this text) creates a new initial value for the ODE at the place where the solution has ‘flowed’ to under \( t \) over an interval of length \( h \). The algorithm restarts at where the solution is at the end of this interval. Because of this the solution is also called a trajectory. We can think of shooting a missile and the path of the missile is the solution.

To obtain a second degree polynomial (or quadratic) approximation to the solution we use \( w = x_0 + x_1t + x_2t^2 \) as the approximation for \( x \). Substituting this into the ODE gives

\[
w' = x_1 + 2x_2t = 192(x_0 + x_1t + x_2t^2) - 256(x_0 + x_1t + x_2t^2)^2 = 192w - 256w^2.
\]

Collecting like powers of \( t \) in this equation leads to

\[
x_1 + 2x_2t = 192x_0 - 256x_0^2 + (192x_1 - 256(2x_0x_1))t + (2x_0x_2 + x_1^2)t^2 + (2x_1x_2)t^3 + x_2^2t^4 = x'(0) + At + Bt^2 + Ct^3 + Dt^4.
\]
From this equation we once again have that $x_1 = x'(0)$, but we also have that $x_2 = \frac{1}{2}(192x_1 - 256(2x_0x_1))$. Since we use $w = x_0 + x_1 t + x_2 t^2$, errors arise in approximating the solution from ignoring the $C$ and $D$ terms. The algorithm to generate a quadratic polynomial approximation to the solution is

(0) Let $w_0 = x_0 = 0.125$ and $k = 1$
(1) Let $x_1 = 192x_0 - 256x_0^2$.
(2) Let $x_2 = \frac{(192x_1 - 256(2x_0x_1))}{2}$.
(3) Let $w(t) = x_0 + x_1 t + x_2 t^2$.
(4) Let $w_k = w(h)$ and then increment $k$ by 1.
(5) Let $x_0 = w_k$ and go to step (1) until $k = K$.

Note that the algorithm for the linear approximation to $x$ is almost the same as the algorithm for the quadratic approximation to $x$. In fact, we can generate a polynomial approximation of any degree for $x$ by using the recurrence relation we obtained above for the $x_{i+1}$ through Cauchy products on the ODE $x' = ax^2 + bx + c$. If one desires an $n^{th}$ degree polynomial approximation to $x$ for the solution to this IV ODE, one modifies the two algorithms above to the following

(0) Let $w_0 = x_0 = 0.125$ and $k = 1$
(1) Let $x_1 = 192x_0 - 256x_0^2$.
(2) Let $x_{i+1} = \frac{1}{i+1}(-256\sum_{j=0}^{i} x_j x_{i-j} + 192x_i)$; $i = 1, 2, 3, ..., n - 1$.
   ($a = -256$, $b = 192$ in the Cauchy product formula above.)
(3) Let $w(t) = x_0 + x_1 t + x_2 t^2 + ... + x_n t^n$.
(4) Let $w_k = w(h)$ and then increment $k$ by 1.
(5) Let $x_0 = w_k$ and go to step (1) until $k = K$.

Since $x = 0.75$ is a stable equilibrium for this ODE, the solution with $x_0 = 0.125$ should eventually approach 0.75. We present the results of approximating the true solution with a line (forward Euler), a quadratic polynomial, a cubic (degree 3) polynomial, a quartic (degree 4) polynomial and a degree 8 polynomial using the algorithms given above. We will use $h = \frac{1}{2^6} = 0.015625$ in our approximating. Since the polynomial approximations are functions of $t$ we can present the approximating values for the true solution at any desired $t$ with $0 < t < Kh$. The figures we present for the solutions will do this, but in certain cases we will also plot the points $(kh, w_k)$.

Figure 1 below is the linear approximation to the solution. As one can see the approximation is not approaching 0.75. The linear approximation is oscillating up and down which is impossible for the solution to do because $x'(t)$ would have to have many zeroes for this to happen, but $x'$ is 0 only at $x = 0$ and $x = 0.75$. The maximum value in Figure 1 is 0.999984 and the minimum value is 0.00022. This approximation does not appear to be a good approximation to the solution. In fact the approximation does not appear to be smooth while the solution is. Mathematicians call the approximation chaotic. We will not consider this topic here, but it is a very interesting one with broad implications. The red points on the graph are the points $(kh, w_k)$. 
In Figure 2 we present the plot of the quadratic approximation to the solution. This approximation jumps to the equilibrium 0.75 at $t = h$ which the true solution can not do. Figure 3 is a plot of the cubic polynomial approximation to the solution. This plot is less accurate than the quadratic polynomial and oscillates like the linear approximation. Figure 4 is a plot of the quartic polynomial approximation to the solution. This approximation passes the equilibrium quickly and grows without bound quickly. Figure 5 is a plot of the degree 8 polynomial approximation to the solution and does approach 0.75 smoothly, but overshoots the equilibrium before it approaches the equilibrium, which is impossible for the solution to do. We could continue the algorithm above to obtain higher degree than 8 polynomial approximations to the solution in order to minimize the overshoot or choose a smaller $h$. 

Figure 1: The linear approximation to the solution of the IV ODE.
Figure 2: The quadratic approximation to the solution of the IV ODE.

Figure 3: The cubic approximation to the solution of the IV ODE.
Figure 4: The quartic approximation to the solution of the IV ODE.

Figure 5: The $8^{th}$ degree approximation to the solution of the IV ODE.
This example shows that just increasing the degree of the polynomial approximation to a solution will not always give better results. However, polynomial approximations of degree greater than or equal to 8 are better approximations to the solution. It is also true that choosing a smaller $h$ will give better approximations to the solution.

In the next sections we will consider more ODEs that are polynomial for which polynomial approximate solutions can be obtained through the use of Cauchy products for polynomials. We will also consider ODEs that can be converted to polynomial ODEs and some of their interesting properties.