DEF

A function (fct) f is a rule that assigns a single element (value) to each element in a set D called the domain of f. The element y (value) f assigns to $x \in D$ is denoted $y = f(x)$. The set of all values R assigned by f is called the range of f. We will write $D(f)$ and $R(f)$ or write

$$f : D \to R$$

to show that f is the function with domain D and range R (or range contained in R).

If the domain of f is the natural numbers or whole numbers, we will also call the range of f a sequence and write the range as

$$\{f(0), f(1), f(2), f(3), f(4), ...\}$$

or

$$\{y_0, y_1, y_2, y_3, y_4, ...\}.$$

where $y_k = f(k)$. *

Notes

1. We will study the range of f, especially for sequences and determine patterns and convergence in these sequences.
2. An algorithm is a sequence of steps that is often described by a function.
3. To determine the values in the range of a function, one has to follow the rule given by the function precisely.

DEF

Let f be a fct

1. A root (or zero) of f is a value r so that $f(r) = 0$.
2. A fixed value of f is a value p so that $f(p) = p$. That is f assigns p to p.
3. The orbit of x_0 under f is the sequence

$$O(x_0;\, f) = \{x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), ...\}$$

This sequence is given by the algorithm

$$x_{k+1} = f(x_k)$$

for $k = 0, 1, 2, 3, ...$

Notes

1. The roots of a function can be found by algebra or approximated by an algorithm.

2. If f is a function from the real numbers into the real numbers, i.e. the domain of f is in the real numbers and the range of f is in the real numbers then a fixed value p of f gives the point $(p, f(p)) = (p, p)$ on the graph of f. That is, the fixed point is also on the line $y = x$.

3. If you program f into a computing device then the orbit of x_0 under f is obtained by entering x_0 and then repeatedly using (pressing the) f (button).

4. If r is a root of f then r is a fixed value of $g(x) = f(x) + x$ and if p is a fixed value of g then p is a root of f. (Show this!)

For example, if $f(x) = \frac{x+6}{x+2}$ then

1. the root of f is given by $x + 6 = 0$ and $f(-6) = 0$ so the root is $r = -6$.

2. if f has a fixed value, it is given by $f(p) = \frac{p+6}{p+2} = p$. Solving this equation, gives $(p+6) = p(p+2)$ or $p^2 + p - 6 = 0$ whose solutions are $p = -3$ and $p = 2$. You should check that $f(2) = 2$ and $f(-3) = -3$. If you graph $y = f(x)$ then the fixed points $(-3, -3)$ and $(2, 2)$ will be on the graph of f and on the line $y = x$. These are the points of intersection of the graph of f and the line $y = x$.

3. The orbit of 0 under f is the sequence

$$O(0; \frac{x+6}{x+2}) = \{0, 3, \frac{9}{5}, \frac{39}{19}, ...\}$$

   This sequence is given by the algorithm

$$x_{k+1} = f(x_k) = \frac{x_k + 6}{x_k + 2}$$

   for $x_0 = 0$ and $k = 0, 1, 2, 3, ...$

NEWTON's Method

Newton's Method is an algorithm that finds a root of a fct f from the reals into the reals. (It can be extended to more complicated fcts.) Newton's Method determines an approximation for the root of f by setting up a fixed point algorithm based on f. One must remember that any equation in one variable can be adjusted algebraically (usually in many ways) to fit into this algorithm. The best choices are those that reach the root (fixed value) in as few iterations (terms of the sequence) as possible.

Suppose we want to solve $y = f(x) = 0$ and f is continuously differentiable with $f'(x) \neq 0$ for x near the root r. The Newton's Method algorithm is then

1. Choose an initial ‘guess’ (value) x_0 for r.
2. Create the function $g(x) = x - \frac{f(x)}{f'(x)}$.
3. Determine the orbit of x_0 under g until the fixed value r of g is determined.

$$x_{k+1} = g(x_k)$$

for $k = 0, 1, 2, 3, ...$ until $f(r)$ is close to 0 or $g(r)$ is close to r.

For example, determine $\frac{1}{\sqrt[3]{25}}$.

We can let $x = \frac{1}{\sqrt[3]{25}}$ and then do some algebra. For example,

$$\frac{1}{x^3} = 25.$$

Now let

$$f(x) = \frac{1}{x^3} - 25.$$

Determining the root of this f gives us $\frac{1}{\sqrt[3]{25}}$. We have that

$$f'(x) = -\frac{3}{x^4}$$

and then the function g for Newton's method is given by

$$g(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\frac{1}{x^3} - 25}{-\frac{3}{x^4}} = x + \frac{x - 25x^4}{3} = \frac{4}{3}x - \frac{25}{3}x^4 = \frac{x}{3}(4 - 25x^3).$$

Note that g is a fairly nice function. The algorithm for determining $\frac{1}{\sqrt[3]{25}}$ using this f is

$$x_{k+1} = g(x_k) = \frac{x_k}{3}\,(4 - 25\,{x_k}^3).$$

If $x_0 = 1$ then $x_1 = g(x_0) = g(1) = \frac{1}{3}(4 - 25 \cdot 1^3) = -7$. Note that 0 is a root and a fixed value for g ($g(0) = 0$).

Algorithm

An algorithm is an ordered process that produces a sequence. For example, Define $S_n = s(n)$ and then let $S_{n+1} = 2\,S_n + 3\,S_{n-1}$. Now build the sequence by

1. Choose initial values for S_0 and S_1.

2. Let $S_{n+1} = 2\,S_n + 3\,S_{n-1}$ for $n = 1, 2, 3, ...$

If $S_0 = 1$ and $S_1 = 2$ then $S_2 = 2\,S_1 + 3\,S_0 = 7$ and $S_3 = 2\,S_2 + 3\,S_1 = 20$, etc. Note that if

$$\frac{S_{k+1}}{S_k} = q$$

for q a real number then

$$\frac{S_k}{S_{k-1}} = q$$

and

$$S_{k+1} = q\,S_k \ ; \ S_k = q\,S_{k-1} \ ; \ S_{k+1} = q\,S_k = q^2\,S_{k-1}$$

and our algorithm $S_{n+1} = 2\,S_n + 3\,S_{n-1}$ is

$$S_{n+1} = q^2\,S_{n-1} = 2\,q\,S_{n-1} + 3\,S_{n-1}$$

so that $q^2 = 2\,q + 3$ if $S_{n-1} \neq 0$.