3. $f(x) = A(\tan^{-1}(x) + B)$

$$\lim_{x \to -\infty} f(x) = A\left(\frac{-\pi}{2} + B\right) = -K$$

$$\lim_{x \to \infty} f(x) = A\left(\frac{\pi}{2} + B\right) = K$$

so $A\left(\frac{\pi}{2} + B\right) = -A\left(\frac{-\pi}{2} + B\right)$

$$\frac{\pi}{2} + B = -\left(\frac{-\pi}{2} + B\right)$$

$$2B = 0 \Rightarrow B = 0$$

so $A\frac{\pi}{2} = K \Rightarrow A = \frac{2}{\pi}K$

$f(x) = \frac{2K}{\pi}\tan^{-1}(x)$ maps $(-\infty, \infty)$ to $(-K, K)$

$f(x) = \frac{2}{\pi}\tan^{-1}(x)$ maps $(-\infty, \infty)$ to $(-1, 1)$

$f(x) = \frac{20}{\pi}\tan^{-1}(x)$ maps $(-\infty, \infty)$ to $(-10, 10)$

Let $y = f^{-1}(x) \Rightarrow f(f^{-1}(x)) = x = f(y)$

$$f(y) = \frac{2K}{\pi}\tan^{-1}(y) = x$$

$$\tan^{-1}(y) = \frac{\pi}{2K}x$$

$$y = \tan\left(\frac{\pi}{2K}x\right)$$

Every interval of the form $(-K, K)$ for $K \in \mathbb{N}$ has the same cardinality.

4(a)

$$\begin{array}{rl}
 & 1 - x + x^2 - x^3 \\ \cline{2-2}
1+x \,\big) & 1 \\
 & 1 + x \\ \cline{2-2}
 & -x \\
- & -x - x^2 \\ \cline{2-2}
 & x^2 \\
- & x^2 + x^3 \\ \cline{2-2}
 & -x^3 \\
- & -x^3 - x^3 \\ \cline{2-2}
 & x^4
\end{array}$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \frac{x^4}{1+x}$$

(b)

$$\begin{array}{rl}
 & \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} \\ \cline{2-2}
x+1 \,\big) & 1 \\
- & 1 + \frac{1}{x} \\ \cline{2-2}
 & -\frac{1}{x} \\
- & -\frac{1}{x} - \frac{1}{x^2} \\ \cline{2-2}
 & \frac{1}{x^2} \\
- & \frac{1}{x^2} + \frac{1}{x^3} \\ \cline{2-2}
 & -\frac{1}{x^3} \\
- & -\frac{1}{x^3} - \frac{1}{x^4} \\ \cline{2-2}
 & \frac{1}{x^4}
\end{array}$$

$$\frac{1}{x+1} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \frac{1}{x^4} + \frac{\frac{1}{x^4}}{x+1}$$

c) $$1-x-x^2\ \overline{\big)\,1}\quad 1+x+2x^2+3x^3+5x^4$$

$$\begin{array}{rl}
 & 1 \\
- & \underline{1-x-x^2} \\
 & x+x^2 \\
- & \underline{x-x^2-x^3} \\
 & 2x^2+x^3 \\
- & \underline{2x^2-2x^3-2x^4} \\
 & 3x^3+2x^4 \\
- & \underline{3x^3-3x^4-3x^5} \\
 & 5x^4+3x^5 \\
- & \underline{5x^4-5x^5-5x^6} \\
 & 8x^5+5x^6
\end{array}$$

$$1-x-x^2 = 1+x+2x^2+3x^3+5x^4+\frac{8x^5+5x^6}{1-x-x^2}$$

(Fibonacci)

d) $$-x^2-x+1\ \overline{\big)\,1}\quad -\frac{1}{x^2}+\frac{1}{x^3}-\frac{2}{x^4}+\frac{3}{x^5}$$

(Fibonacci)

$$\begin{array}{rl}
 & 1 \\
- & \underline{1+\frac{1}{x}-\frac{1}{x^2}} \\
 & -\frac{1}{x}+\frac{1}{x^2} \\
- & \underline{-\frac{1}{x}-\frac{1}{x^2}+\frac{1}{x^3}} \\
 & \frac{2}{x^2}-\frac{1}{x^3} \\
 & \underline{\frac{2}{x^2}+\frac{2}{x^3}-\frac{2}{x^4}} \\
 & -\frac{3}{x^3}+\frac{2}{x^4} \\
 & \underline{-\frac{3}{x^3}-\frac{3}{x^4}+\frac{3}{x^5}} \\
 & \frac{5}{x^4}-\frac{3}{x^5}
\end{array}$$

$$\frac{1}{-x^2-x+1} = \frac{-1}{x^2}+\frac{1}{x^3}-\frac{2}{x^4}+\frac{3}{x^5}+\frac{\frac{5}{x^4}-\frac{3}{x^5}}{-x^2-x+1}$$

5. $G_0 = a, \; G_1 = b$

$$G_{K+1} = G_K + G_{K-1}$$

$$G_2 = G_1 + G_0 = b + a$$

$$G_3 = G_2 + G_1 = (b+a) + b = 2b + a$$

$$G_4 = G_3 + G_2 = (2b+a) + (b+a) = 3b + 2a$$

$$G_5 = G_4 + G_3 = (3b+2a) + (2b+a) = 5b + 3a$$

$$G_6 = G_5 + G_4 = (5b+3a) + (3b+2a) = 8b + 5a$$

Coefficients of b and a are Fibonacci sequence

$$\frac{G_{K+1}}{G_K} = 1 + \frac{G_{K-1}}{G_K}$$

Suppose $\lim_{K\to\infty} \frac{G_{K+1}}{G_K} = r$ then $\lim_{K\to\infty} \frac{G_{K-1}}{G_K} = \frac{1}{r}$

so $r = 1 + \frac{1}{r}$

$$r^2 = r + 1$$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$r > 0$ so $r = \frac{1+\sqrt{5}}{2}$