Consider the non-homogeneous linear ODE

$$y'' + y = f(t). \tag{1}$$

We look for solutions to this ODE using variation of parameters. With this choice, we assume the solution is

$$y(t) = u_1(t)\cos t + u_2(t)\sin t$$

since $\{\cos t, \sin t\}$ is a set of fundamental solutions to $y'' + y = 0$. Differentiating y we obtain

$$y' = -u_1 \sin t + u_2 \cos t + u_1' \cos t + u_2' \sin t.$$

Now assuming

$$u_1' \cos t + u_2' \sin t = 0 \tag{2}$$

and differentiating again gives us

$$y'' = -u_1 \cos t - u_2 \sin t - u_1' \sin t + u_2' \cos t.$$

Substituting this into Equation (1) leads to

$$y'' + y = -u_1' \sin t + u_2' \cos t = f(t).$$

Equation (2) and this last equation gives the system of equations

$$\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

Using the inverse of the matrix on the left hand side gives one

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

or

$$\begin{aligned} u_1' &= -f(t)\,\sin t \\ u_2' &= f(t)\,\cos t \end{aligned}$$

Integrating these two equations leads to

$$u_1(t) = u_1(0) - \int_0^t f(\tau)\ \sin\tau\, d\tau$$
$$u_2(t) = u_2(0) + \int_0^t f(\tau)\ \cos\tau\, d\tau$$

Since $y(t) = u_1 \cos t + u_2 \sin t$, we now have

(3) $$y(t) = u_1(0)\ \cos t - \cos t \int_0^t f(\tau)\ \sin\tau\, d\tau + u_2(0)\ \sin t + \sin t \int_0^t f(\tau)\ \cos\tau\, d\tau$$

One can write this last form for y as

$$y(t) = u_1(0)\ \cos t + u_2(0)\ \sin t + \int_0^t f(\tau)\ (\cos\tau \sin t - \sin\tau \cos t)\ d\tau$$

or

$$y(t) = u_1(0)\ \cos t + u_2(0)\ \sin t + \int_0^t f(\tau)\ \sin(t-\tau)\, d\tau.$$

This last form is known as the Green's function form for the solution. Since $u_1(0), u_2(0)$ are arbitrary, we write this last solution as

$$y(t) = c_1\ \cos t + c_2\ \sin t + \int_0^t f(\tau)\ \sin(t-\tau)\, d\tau.$$

Therefore, the solution is the homogenous part plus a convolution of the forcing term with sin.

If we let

$$v = y',\ A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } M = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

then $y'' + y = 0$ can be written as the equivalent system

$$x' = Ax$$

with $x = \begin{pmatrix} y \\ v \end{pmatrix}$ whose solution is $x_H = M \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ where c_1 and c_2 are arbitrary numbers.

We now look for a solution to the non-homogeneous problem

$$x' = Ax + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

which is equivalent to $y'' + y = f(t)$ by assuming

$$x_N = Mu = M \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

Differentiating gives

$$x_N' = Mu' + M'u.$$

Substituting x_N into the system of ODEs leads to

$$x_N' = Mu' + M'u = Ax_N + \begin{pmatrix} 0 \\ f(t) \end{pmatrix} = AMu + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

Since $M' = AM$, this simplifies to

$$Mu' = \begin{pmatrix} 0 \\ f(t) \end{pmatrix} \equiv F(t).$$

Therefore, $u' = M^{-1}F(t)$. The determinant of M is the Wronskian of $\{\cos t, \sin t\}$ and is equal to 1. The inverse of M is

$$M^{-1} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Therefore,

$$u' = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} 0 \\ f(t) \end{pmatrix}$$

giving

$$u_1' = -f(t)\,\sin t$$
$$u_2' = f(t)\,\cos t$$

which is exactly what we obtained above using $y'' + y = f(t)$.