Chapter 2.

Egg1:
The Quadratic Ordinary Differential Equation

Goal: Just as moving from linear to nonlinear in any part of mathematics introduces more complexity, this chapter presents the concepts of power series solutions to ordinary differential equations with a quadratic right hand side. The concept of Cauchy product may be difficult for students to understand, so make sure you calculate as many of the terms as needed in order to convince yourselves of the algorithms.
Mathematical Prerequisites: Understanding of the basic concepts of solving ordinary differential equations, including equilibrium solutions.
Additional thoughts: This chapter is more advantageous if students compute solutions using their knowledge, not just replicating what they have seen before. The work of Knuth is referenced in this chapter.

This chapter will focus on the nonlinear ODE

$$x' = P_2(x)$$

where the RHS, P_2, is the quadratic polynomial given by

$$P_2(x) = ax^2 + bx + c. \tag{1}$$

The addition of the term ax^2 to the RHS of the linear ODE presented in Chapter 1, makes this ODE nonlinear (The graph of the RHS, $P_2(x)$, is a parabola not a line.), more interesting, and more complicated. However, most of the techniques presented in the last chapter will aid us in learning about this ODE.

As a simple case, consider the ODE

$$x' = x^2;\ x(0) = x_0,$$

where $a = 1, b = 0, c = 0$ from (**??**). Proceeding as in the last chapter, substitute the polynomial

$$x = \sum_{i=0} x_i t^i$$

into this ODE and get

$$x' = \sum_{i=0} (i+1) x_{i+1} t^i = \left(\sum_{i=0} x_i t^i \right)^2 .$$

Cauchy (see Knuth) showed that for two convergent power series

$$\sum_{i=0} a_i t^i \text{ and } \sum_{i=0} b_i t^i,$$

the product of these two power series is given by

$$\begin{aligned}\sum_{i=0} a_i t^i \sum_{i=0} b_i t^i &= (a_0 + a_1 t + a_2 t^2 + ...)(b_0 + b_1 t + b_2 t^2 + ...) \\ &= a_0 b_0 + (a_0 b_1 + a_1 b_0)t + (a_0 b_2 + a_1 b_1 + a_2 b_0)t^2 + ... \\ &= \sum_{i=0} \left(\sum_{j=0}^{i} a_k b_{j-i} \right) t^i,\end{aligned}$$

which will be hereafter referred to as a *Cauchy product.*

Applying the Cauchy product to $\left(\sum_{i=0} x_i t^i\right)^2$ above where $a_i = x_i = b_i$ for $i = 0..k$ gives

$$\begin{aligned}\left(\sum_{i=0} x_i t^i\right)^2 = \sum_{i=0} x_i t^i \sum_{i=0} x_i t^i &= (x_0 + x_1 t + x_2 t^2 + ...)(x_0 + x_1 t + x_2 t^2 + ...) \\ &= x_0^2 + 2x_0 x_1 t + (2x_0 x_2 + x_1^2)t^2 + ... \\ &= \sum_{i=0} \left[\sum_{j=0}^{i} x_j x_{i-j} \right] t^i = x_0^2 + \sum_{i=1} \left[\sum_{j=0}^{i} x_j x_{i-j} \right] t^i.\end{aligned}$$

Substituting the above expression into the ODE $x' = x^2$ leads to

$$\begin{aligned}x' &= x_1 + 2x_2 t + 3x_3 t^2 + ... + (k+1)x_{k+1} t^k \\ &= x^2 \\ &= x_0^2 + 2x_0 x_1 t + (2x_0 x_2 + x_1^2)t^2 + ... + \left[\sum_{j=0}^{k} x_j x_{k-j} \right] t^k\end{aligned}$$

Equating the coefficients of equal powers of t and solving for x_i gives

$$\begin{aligned}x_1 &= x_0^2, \\ x_2 &= x_0 x_1 = x_0^3, \\ x_3 &= \frac{1}{3}(2x_0 x_2 + x_1^2) = x_0^4, \\ &. \\ &. \\ &. \\ x_i &= \frac{1}{i} \sum_{j=0}^{i-1} x_j x_{i-j} = x_0^{i+1} \ ; \ i = 2, 3, 4, ...\end{aligned}$$

Using this above recurrence relationship, we determine that

$$x = x_0 + x_0^2 t + x_0^3 t^2 + ... + x_0^{k+1} t^k = x_0(1 + x_0 t + (x_0 t)^2 + ... + (x_0 t)^k) = x_0 \sum_{j=0}^{k} (x_0 t)^j.$$

We note that this is a geometric series and converges to

$$x = x_0 \frac{1}{1 - x_0 t}$$

as $k \to \infty$. Note that $x(0) = x_0$ and

$$x' = x_0^2 (1 - x_0 t)^{-2} = x^2.$$

Therefore, the Maclaurin series found using Cauchy products converges to the solution of the IV ODE.

We now consider the equivalent integral equation for this IV ODE, which is

$$x(t) = x_0 + \int_0^t x(\tau)^2 \, d\tau.$$

The Picard Iterates are

$$\begin{aligned} p_0(t) &= x(0) = x_0 \\ p_{k+1}(t) &= x_0 + \int_0^t (p_k(\tau))^2 \, d\tau \text{ for } k = 0, .., n. \end{aligned}$$

The first few Picard Iterates are

$$\begin{aligned} p_0(t) &= x_0 \\ p_1(t) &= x_0 + \int_0^t (x_0)^2 \, d\tau \ = x_0 + x_0^2 \, t \\ p_2(t) &= x_0 + \int_0^t (p_1(\tau))^2 \, d\tau \ = x_0 + \int_0^t (x_0 + x_0^2 \, \tau)^2 \, d\tau \\ &= x_0 + x_0^2 \, t + x_0^3 \, t^2 + x_0^4 \, \frac{t^3}{3} \\ p_3(t) &= x_0 + \int_0^t (p_2(\tau))^2 \, d\tau \ = \\ &= x_0 + x_0^2 \, t + x_0^3 \, t^2 + {x_0}^4 t^3 + \frac{2}{3} \, {x_0}^5 t^4 + \frac{1}{3} \, {x_0}^6 t^5 + \frac{1}{9} \, {x_0}^7 t^6 + \frac{1}{63} \, {x_0}^8 t^7. \end{aligned}$$

Observe that the number of terms in the Picard iterates will increase by a factor of 2 over the previous iteration. Therefore, the number of terms will become unwieldy after only a few iterations. Also notice that the first j terms in p_j agree with the first j terms of the j^{th} degree Maclaurin polynomial.

Changing p_2 to

$$p_2(t) = x_0 + x_0^2\,t + x_0^3\,t^2$$

and defining

$$p_3(t) = x_0 + \int_0^t (p_2(\tau))^2\; d\tau$$

gives

$$p_3(t) = x_0 + x_0^2\,t + x_0^3\,t^2 + x_0^4\,t^3 + \text{ higher order terms in } t.$$

Now redefine

$$p_3(t) = x_0 + x_0^2\,t + x_0^3\,t^2 + x_0^4\,t^3$$

which is the third degree Maclaurin polynomial approximation to the solution x of the IV ODE and the truncation of the previous p_3 to degree 3. Continue this process of dropping all the terms containing a t^k with $k > j$ then the redefined p_j would be the j^{th} degree Maclaurin polynomial for x.

Parker and Sochacki showed that this equality was true for all IV ODEs with polynomial RHS. This fact will be used in the later chapters for other nonlinear ODEs, but in most cases we will generate Maclaurin polynomial approximations to the solution x of the IV ODE using Cauchy products as presented earlier for this IV ODE.

Consider the following example from Knuth. Recall that determining the Maclaurin polynomial approximation to the function $x = \tan t$ in calculus is a complicated assignment. In fact, if $x = \tan t$ then

$$\begin{aligned} x' &= \sec^2 t, \\ x'' &= 2\sec^2 t \tan t, \\ x''' &= 4\sec^2 t \tan^2 t + 2\sec^4 t, \end{aligned}$$

and the higher derivatives are even more complicated to calculate.

Knuth notes that if $h(t) = \frac{f(t)}{g(t)}$ with

$$f(t) = \sum_{i=0} f_i t^i;\;\; g(t) = \sum_{i=0} g_i t^i;\;\; h(t) = \sum_{i=0} h_i t^i$$

then $f(t) = g(t)h(t)$ and

$$f(t) = \sum_{i=0} f_i t^i = g(t)h(t) = \sum_{i=0} g_i t^i \ \sum_{i=0} h_i t^i.$$

Applying Cauchy's rule for the product of two power series given above to $f(t)$ gives

$$\begin{aligned} f(t) = \sum_{i=0} g_i t^i \ \sum_{i=0} h_i t^i &= (g_0 + g_1 t + g_2 t^2 + ...)(h_0 + h_1 t + h_2 t^2 + ...) \\ &= g_0 h_0 + (g_0 h_1 + g_1 h_0)t + (g_0 h_2 + g_1 h_1 + g_2 h_0)t^2 + ... \\ &= \sum_{i=0} \left[\sum_{j=0}^{i} g_j h_{i-j} \right] t^i = g_0 h_0 + \sum_{i=1} \left[\sum_{j=0}^{i} g_j h_{i-j} \right] t^i. \end{aligned}$$

Equating the coefficients of equal powers of t for $f(t)$ with $g(t)h(t)$ and solving for the coefficients of h gives

$$\begin{aligned} h_0 &= \frac{f_0}{g_0}, \\ h_i &= \frac{1}{g_0} \left(f_i - \sum_{j=0}^{i-1} g_{j+1} h_{i-1-j} \right) \ ; \ i = 1, 2, 3, ... \end{aligned}$$

Since $x = h(t) = \frac{\sin t}{\cos t}$, we can obtain the power series for $\tan t$ using the above recurrence relation with the power series for $f(t) = \sin t$ and $g(t) = \cos t$. We have

$$\begin{aligned} f(t) &= \sin t = t - \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{7!}t^7 + ..., \\ g(t) &= \cos t = 1 + \frac{1}{2}t^2 - \frac{1}{4!}t^4 + ..., \end{aligned}$$

giving $f_{2i} = 0, \quad f_{2i+1} = (-1)^i \frac{1}{(2i+1)!}$, $g_{2i} = (-1)^i \frac{1}{(2i)!}$ and $g_{2i+1} = 0$ for $i = 0, 1, 2, 3, ...$ Using this information leads to

$$\begin{aligned} \sum_{i=0} h_i t^i &= \frac{\sum_{i=0} \frac{1}{(2i+1)!} t^{2i+1}}{\sum_{i=0} \frac{1}{(2i)!} t^{2i}}, \\ h_0 &= \frac{f_0}{g_0} = \frac{\sin(0)}{\cos(0)} = 0, \\ h_{2i} &= \frac{1}{g_0} \left(f_{2i} - \sum_{j=0}^{2i-1} g_{j+1} h_{2i-1-j} \right) = 0 \ ; \ i = 1, 2, 3, ..., \\ h_{2i-1} &= \frac{1}{g_0} \left(f_{2i-1} - \sum_{j=0}^{2i-2} g_j h_{2i-2-j} \right) = (-1)^{i-1} \frac{1}{(2i-1)!} - \sum_{j=0}^{2i-2} g_j h_{2i-2-j} \ ; \ i = 1, 2, 3, ... \end{aligned}$$

Since $\tan t$ is an odd function, $h_{2i} = 0$ for all i. We find for the first few odd indices of h using the above recurrence relation that $h_1 = 1, h_3 = \frac{2}{15}, h_5 = \frac{17}{315}$. As many coefficients as are desired for $\tan t$ can be determined by continuing this recurrence relation.

Another way to determine a power series for $x = h(t) = \frac{f(t)}{g(t)}$, is to differentiate x and use $w = \frac{1}{g(t)}$. That is,

$$x' = \frac{f'g - fg'}{g^2} = (f'g - fg')w^2$$
$$w' = -w^2 g'.$$

Notice that because $f = \sin t$ and $g = \cos t$ the above leads to the sytem of ODEs

$$x' = (f^2 + g^2)w^2 = w^2$$
$$w' = w^2 f$$
$$f' = g$$
$$g' = -f.$$

Substituting $z = wf$ in these equations gives

$$z' = w'f + wf' = w^2 f^2 + wg = z^2 + 1.$$

The last equality arises from the fact that $z = wf$ and $w = \frac{1}{g}$.

Since $z = \tan t$ satisfies the IV ODE

$$z' = z^2 + 1 \; ; \; z(0) = 0,$$

one notices that $x = \tan t$ satisfies $x' = 1 + x^2$. We now substitute the power series $x = \sum x_i t^i$ into this IV ODE for x. (Note that the RHS for this ODE is a quadratic polynomial in x) to obtain the power series for $\tan t$. Doing this gives

$$x' = \sum_{i=0}^{\infty}(i+1)x_{i+1}t^i = 1 + x^2 = 1 + (\sum_{i=0}^{\infty} x_i t^i)^2 = 1 + x_0^2 + \sum_{i=1}^{\infty}\left[\sum_{j=0}^{i} x_j x_{i-j}\right] t^i.$$

We obtained the last equality by again using Cauchy products for power series. Equating the coefficients of like powers of t on both sides of the equation leads to $x_1 = 1 + x_0^2$ and

$$x_{i+1} = \frac{1}{i+1}\sum_{j=0}^{i} x_j x_{i-j} \text{ for } i = 1, 2, 3...$$

Since $x_0 = \tan 0 = 0$, we have $x_1 = 1$ and using the above recurrence relation gives us all the other coefficients in the power series for $\tan t$.

The Quadratic ODE

We now generalize and extend the method presented above to determine power series solutions for ODEs with quadratic polynomial RHS, $ax^2 + bx + c$. That is, consider the IV ODE

$$x' = ax^2 + bx + c \; ; \; x(0) = x_0.$$

If $a = 0$ we have the linear ODE we studied in the last chapter. For $a = 1, b = 0, c = 0$ and $a = 1, b = 0, c = 1, x_0 = 0$ we have the ODEs for $x = \frac{x_0}{1-x_0 t}$ and $x = \tan t$, respectively that we studied above. We can determine the power series for this general quadratic ODE in the same manner as these two quadratic ODEs.

Substitute $x = \sum x_i t^i$ into the ODE and then uses Cauchy products for power series to obtain

$$\begin{aligned} x' = \sum_{i=0}^{\infty}(i+1)x_{i+1}t^i &= ax^2 + bx + c \\ &= a\left(\sum_{i=0}^{\infty} x_i t^i\right)^2 + b\sum_{i=0}^{\infty} x_i t^i + c \\ &= ax_0^2 + bx_0 + c + a\sum_{i=1}^{\infty}\left[\sum_{j=0}^{i} x_j x_{i-j}\right] t^i + b\sum_{i=1}^{\infty} x_i t^i \\ &= ax_0^2 + bx_0 + c + \sum_{i=1}^{\infty}\left[a\sum_{j=0}^{i} x_j x_{i-j} + bx_i\right] t^i \end{aligned}$$

We ask the reader to show that the recurrence relation obtained from these substitutions by equating coefficients of like powers of t for x' in the last equation on the right above is

$$x_1 = ax_0^2 + bx_0 + c$$

$$x_{i+1} = \frac{1}{i+1}(a\sum_{j=0}^{i} x_j x_{i-j} + bx_i); \; i = 1, 2, 3, ...$$

This recurrence relation gives the coefficients for the power series of x. We point out that a simple recurrence like the above is straightforward to obtain using Cauchy products for power series for all quadratic polynomials systems of IV ODEs. We will demonstrate this more generally later in this work.

The above recurrence relation gives

$$x = x_0 + (ax_0^2 + bx_0 + c)t + \left(a^2{x_0}^3 + a\left(cx_0 + \frac{3}{2}\,b{x_0}^2\right) + \frac{1}{2}\,b^2x_0 + \frac{1}{2}\,bc\right)t^2 +$$
$$\left(a^3{x_0}^4 + a^2\left(\frac{4}{3}\,c{x_0}^2 + 2b\,{x_0}^3\right) + \left(\frac{1}{3}\,c^2 + \frac{4}{3}\,bcx_0 + \frac{7}{6}\,b^2{x_0}^2\right)a + \frac{1}{6}\,b^3x_0 + \frac{1}{6}\,b^2c\right)t^3 +$$
$$\ldots$$

Chapter 1 examined the linear ODE $x' = bx + c$ and considered the power series solution for this ODE. From the power series solution above, notice the impact ax^2 has on the two solutions. Substitute $a = 0$ in the above power series in order to obtain the power series in the last chapter for linear ODEs.

Even if a and x_0 are close to 0, it can be seen that as t gets close to 1 and then greater than 1 the above power series solution diverges from the power series solution given in Chapter 1 for $x' = bx + c;\ x(0) = x_0$. One can also observe that increasing or decreasing the parameter a significantly changes the value of the coefficients in the power series. Having the power series solution allows one to analyze the impact of terms involving the dependent variable, coefficients and parameters in the ODE even for non-polynomial ODEs as will be shown in later chapters. We will demonstrate this some in numerical examples below and in other chapters.

It is seen in the first two examples of this chapter that the properties of the solutions to the ODE change drastically as the right hand side of the solution is changed from a line $(bx + c)$ to a quadratic polynomial (parabola $ax^2 + bx + c$) no matter the magnitude of a. We now outline why this is the case.

We now consider how a changes the general form of the solution from the case of the linear ODE in terms of the equilibrium solution. Remember from Chapter 1 that the linear ODE has one and only one equilibrium solution. Recall also that the coefficient b in the linear ODE determines whether the solution approaches the equilibrium or grows or decays without bound. We will outline the three possible cases for the quadratic IV ODE. For completeness, we present the solution of the ODE for each of these three cases.

Case 1. No roots: $x' = ax^2 + bx + c = a((x - r_1)^2 + r_2^2)$

Since $ax^2 + bx + c$ has no real roots, the above ODE has no equilibrium solutions. The RHS can be written as the sum of two squares with real numbers r_1 and r_2 (which compose the real and imaginary part of the complex roots, $r_1 \pm i\, r_2$ for $ax^2 + bx + c = 0$) as shown in the last equality for the ODE above. When the ODE is expressed in this form it can be integrated to give

$$x = r_1 + \tan\ ar_2^2(t + C).$$

Case 2. One root: $x' = ax^2 + bx + c = a(x - r)^2$

Since $ax^2 + bx + c$ has only one real root, r it can be written as a perfect square. There is exactly one equilibrium solution. The solution for this ODE is

$$x = \frac{ar\ t + rC - 1}{at + C}.$$

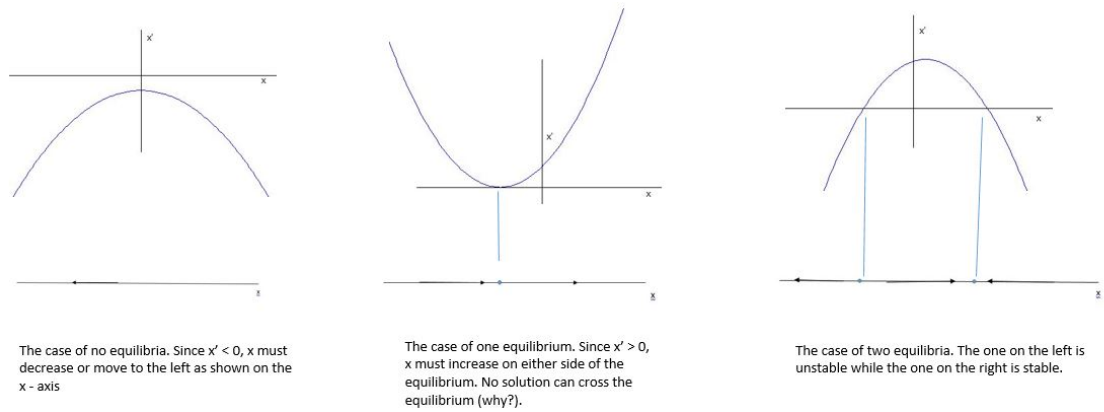

Figure 1: A graph of the types of equilibria for an ODE with quadratic polynomial on the RHS.

Case 3. $x' = ax^2 + bx + c = a(x - r_1)(x - r_2)$

Since $ax^2 + bx + c$ has two real roots, r_1, r_2 it can be factored into these roots. This ODE has two equilibrium solutions. The solution for this ODE is

$$x = \frac{r_2 e^{aC(r_1 - r_2)\ t + C(r_1 - r_2)} - r_1}{e^{aC(r_1 - r_2)\ t + C(r_1 - r_2)} - 1}.$$

where C is determined by the IV $x(0) = x_0$ in all three cases.

In Figure 1 we show a typical graph of x' vs x for each of these cases that highlights the equilibria and their types. We also have a graph of just x showing the direction a solution must move on the x - axis. We leave it for the reader to draw graphs of x vs t for each of the three cases.

We summarize what the quadratic ODE gives us and note that it is significantly different from the linear ODE. The linear ODE has basically one equilibrium condition that determines the type of all solutions. Since the quadratic equation $ax^2+bx+c = 0$ has zero, one or two real roots, there are three basic types of solutions for the ODE $x' = ax^2 + bx + c$. For the case with no real roots the parabola $y = ax^2 + bx + c$ lies completely above or below the x-axis. Since $y = x'$, in the first case the solution to this ODE is always increasing and in the second case the solution is always decreasing. For the case of one real root the solution to this ODE is again always increasing or always decreasing since the graph of y lies above or below the x-axis except at the vertex of the parabola which lies on the x-axis. Since the solutions to this quadratic ODE are unique, no solution can cross the equilibrium. Also, since at least one solution will always diverge from the equilibrium in this case, the equilibrium is unstable. For the case of two real roots the parabola lies both above and below the x - axis. Therefore, solutions close to one equilibrium approach the equilibrium and solutions close to the other equilibrium diverge away from the equilibrium. This means one equilibrium solution is stable and the other is unstable.

Another significant difference from linear ODEs is that all solutions 'blow up in finite t'. That is, there is a real number t_* (it may be negative) so that $\lim_{t \to t_*} x = \pm\infty$. Therefore, the power series given above for the solution for all three of these cases has a finite radius of convergence as opposed to the linear case where the power series converges for all t.

In the next example we will demonstrate that solutions are drastically different from those ODEs whose right hand side is slightly different from a line. At the end of this section we will show why this is the case.

In the next chapters, we will consider more ODEs that are polynomial for which polynomial approximate solutions can be obtained through the use of Cauchy products for polynomials. We will also consider ODEs that can be converted to polynomial ODEs and some of their interesting properties. In the project section we look at a famous population example quadratic IV ODE (see Strogatz or Braun) and its solution using power series.

Projects, Questions, and Future Consideration

To generate projects and questions we will work through a population example in detail. In studying the dynamics of populations of species quadratic IV ODEs describing the dynamics of the population are appropriate.

We will generate polynomials that are approximations to the power series solution for a changing population that is governed by a quadratic polynomial IV ODE using Cauchy products as above. We will demonstrate that these polynomials can give us an accurate solution, but that care must be taken in determining the degree of these polynomials and the interval of convergence of the power series solutions. We will also make use of the fact that we can always repose the IV ODE at a point on the solution as we demonstrated in Chapter 1. At the end of the chapter we will ask you to do the same analysis on two questions.

If $x(t)$ represents the number of individuals in the population that competes among its own members then it reasonable to assume that x satisfies the IV ODE

$$x' = \beta x - \alpha x^2 = x(\beta - \alpha x); \;\; x(0) = x_0.$$

The equilibrium solutions for this ODE are $x = 0$ and $x = \frac{\beta}{\alpha}$ (calculated by setting $x' = 0$). From the discussion above, one equilibrium is unstable and the other one is stable depending on $a = -\alpha, b = \beta$. This makes sense in population dynamics because the population could die out or it could approach a livable equilibrium. (If $x_0 = 0$ there is nothing to study since the population will always be 0 in this case.) For example, if α, β are both positive then $x = 0$ is an unstable equilibrium and $x = \frac{\beta}{\alpha}$ is a stable equilibrium. Therefore, in this case the population will tend toward $\frac{\beta}{\alpha}$.

We now look at the specific case where $a = 256, b = 192$. We assume that $x = 1$ represents one thousand individuals ($x = 2$ represents two thousand individuals) and so on. The stable equilibrium in this case is $x = \frac{\beta}{\alpha} = \frac{3}{4}$ representing a population of 750. The IV ODE is

$$x' = 192x - 256x^2 = x(192 - 256x);\; x(0) = x_0 = 0.125.$$

That is, we are assuming that there are initially 125 individuals living under this population process.

From above, we know the closed form (analytic) solution to this IV ODE. We could just plot that solution and see its graph approach the equilibrium $\frac{3}{4}$. We point out that software packages generate the graph by plotting points $(t, x(t))$ by plugging values for a variety of t's into the solution. The software package has to do some numerical approximation to determine a numerical value for $x(t)$. Many software packages use some power series scheme to determine these approximations. We, instead, will assume that the solution is a polynomial and see if we can approximate the power series solution accurately by using the Cauchy product algorithm we developed in this chapter.

First, let us assume that $w = x_0 + x_1 t$ is the solution to this problem. Then

$$w' = x_1 = 192w - 256w^2 = 192(x_0 + x_1 t) - 256(x_0 + x_1 t)^2.$$

Collecting like powers of t in this equations leads to

$$\begin{aligned} x_1 &= 192x_0 - 256x_0^2 + (192x_1 - 256(2x_0x_1))t - 256x_1^2t^2 \\ &= x'(0) + At + Bt^2. \end{aligned}$$

Therefore, we should set $x_1 = x'(0) = 192x_0 - 256x_0^2 = 1$. However, performing this substitution ignores the A and B terms arising from the ODE. This will lead to an error in using $w = x_0 + x_1 t = 0.125 + t$ as the solution. Therefore, we call this w an approximation to the solution x.

We now use this w to generate an approximation to the solution x at various t. In particular, we will do this at the t in the set $\{t_k | t_k = kh,\; k \in \mathbb{N}\}$ (here $\mathbb{N}$ represents the set of natural numbers 1,2,3,...) where h is some fixed positive real number and will be called the time step for the approximation to the solution. We will also call this approximation the approximate solution or the numerical solution.

Here is the algorithm to obtain these approximations.

(0) Let $w_0 = x_0 = 0.125$ and $k = 1$
(1) Let $x_1 = 192x_0 - 256x_0^2$.
(2) Let $w(t) = x_0 + x_1 t$.
(3) Let $w_k = w(h)$ and then increment k by 1.
(4) Let $x_0 = w_k$ and go to step (1) until $k = K$.

This algorithm is known as the forward Euler approximation to the solution of the IV ODE. It is a first degree polynomial (or linear) approximation to the solution to the IV ODE.

The updating of x_0 in the algorithm above (and all the ones presented in this text) creates a new initial value for the ODE at the place where the solution has 'flowed' to under t over an interval of length h. The algorithm restarts at where the solution is at the end of this interval. Because of this, the solution is also called a trajectory. We can think of shooting a missile and the path of the missile is the solution.

To obtain a second degree polynomial (or quadratic) approximate solution, we use $w = x_0 + x_1 t + x_2 t^2$ as the approximation for x. Substituting this into the ODE gives

$$w' = x_1 + 2x_2 t = 192(x_0 + x_1 t + x_2 t^2) - 256(x_0 + x_1 t + x_2 t^2)^2 = 192w - 256w^2.$$

Collecting like powers of t in this equation leads to

$$\begin{aligned} & x_1 + 2x_2 t \\ & = 192x_0 - 256x_0^2 + (192x_1 - 256(2x_0 x_1))t + (2x_0 x_2 + x_1^2)t^2) + (2x_1 x_2)t^3 + x_2^2 t^4 \\ & = x'(0) + At + Bt^2 + Ct^3 + Dt^4. \end{aligned}$$

From this equation we once again have that $x_1 = x'(0) = 1$, but we now have that $x_2 = \frac{1}{2}(192x_1 - 256(2x_0 x_1)) = -20$. Since we use $w = x_0 + x_1 t + x_2 t^2$ to approximate the solution, errors arise in this approximate solution from ignoring the C and D terms. The algorithm to generate a quadratic polynomial approximation to the solution is

(0) Let $w_0 = x_0 = 0.125$ and $k = 1$
(1) Let $x_1 = 192x_0 - 256x_0^2$.
(2) Let $x_2 = \frac{(192x_1 - 256(2x_0 x_1))}{2}$.
(3) Let $w(t) = x_0 + x_1 t + x_2 t^2$.
(4) Let $w_k = w(h)$ and then increment k by 1.
(5) Let $x_0 = w_k$ and go to step (1) until $k = K$.

Note that the algorithm for the linear approximation to x is almost the same as the algorithm for the quadratic approximation to x. In fact, we can generate a polynomial approximation of any degree for x by using the recurrence relation we obtained above for the x_{i+1} through Cauchy products on the ODE $x' = ax^2 + bx + c$. If one desires an n^{th} degree polynomial approximation to x for the solution to this IV ODE, one modifies the two algorithms above to the following

(0) Let $w_0 = x_0 = 0.125$ and $k = 1$
(1) Let $x_1 = 192x_0 - 256x_0^2$.
(2) Let $x_{i+1} = \frac{1}{i+1}(-256 \sum_{j=0}^{i} x_j x_{i-j} + 192x_i)$; $i = 1, 2, 3, ..., n-1$.
($a = -256$, $b = 192$ in the Cauchy product formula above.)
(3) Let $w(t) = x_0 + x_1 t + x_2 t^2 + ... + x_n t^n = \sum_{i=0}^{n} x_i t^i$.
(4) Let $w_k = w(h)$ and then increment k by 1.
(5) Let $x_0 = w_k$ and go to step (1) until $k = K$.

Since $x = 0.75$ is a stable equilibrium for this ODE, the solution with $x_0 = 0.125$ should eventually approach 0.75. We present the results of approximating the true solution with a line (forward Euler), a quadratic polynomial, a cubic (degree 3) polynomial, a quartic (degree 4) polynomial and a degree 8 polynomial using the algorithms given above. We will use $h = \frac{1}{2^6} = 0.015625$ in the approximation. Since the polynomial approximations are functions of t, we can present the approximating values for the true solution at any desired t with $0 < t < Kh$. The figures present the solutions, but in certain cases the points (kh, w_k) are also plotted.

Figure 2 below is the linear approximation to the solution. Notice that the approximation is not approaching 0.75. The linear approximation is oscillating up and down, which is impossible for the solution to do because $x'(t)$ would have to have many zeroes for this oscillation to occur, but x' is 0 only at $x = 0$ and $x = 0.75$. The maximum value in Figure 2 is 0.999984 and the minimum value is 0.00022. This approximation does not appear to be a good approximation to the solution. In fact the approximation does not appear to be smooth while the solution is. Mathematicians call the approximation chaotic. We will not consider this topic here, but it is a very interesting one with broad implications. The red points on the graph are the points (kh, w_k).

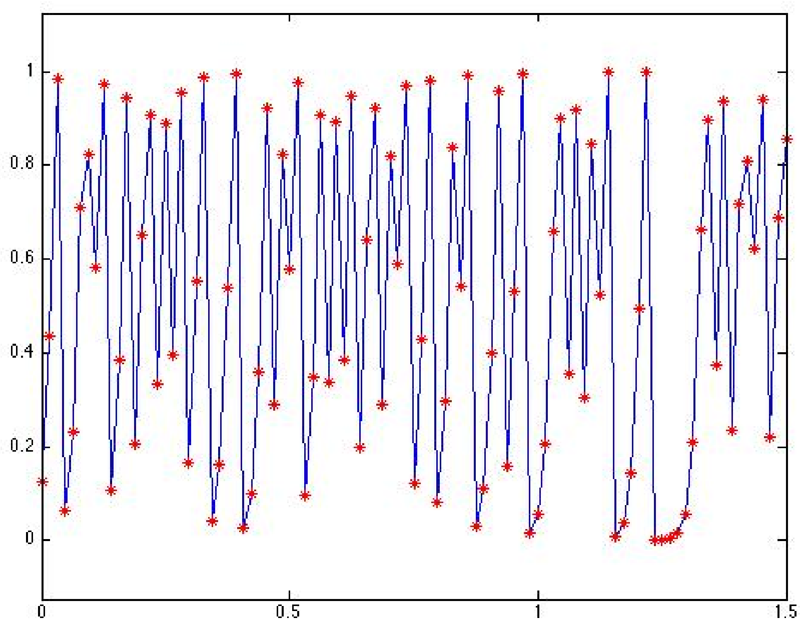

Figure 2: The linear approximation to the solution of the IV ODE.

Figure 3 presents the plot of the quadratic approximation to the solution. This approximation jumps to the equilibrium 0.75 at $t = h$ which the true solution cannot do. Figure 4 is a plot of the cubic polynomial approximation to the solution. This plot is less accurate than the quadratic polynomial and oscillates like the linear approximation. Figure 5 is a plot of the quartic polynomial approximation to the solution. This approximation passes the equilibrium quickly and grows without bound quickly. Figure 6 is a plot of the degree 8 polynomial approximation to the solution and does approach 0.75 smoothly, but overshoots the equilibrium before it approaches the equilibrium, which is impossible for the solution to do. We could continue the algorithm above to obtain higher degree than 8 polynomial approximations to the solution in order to minimize the overshoot or choose a smaller h.

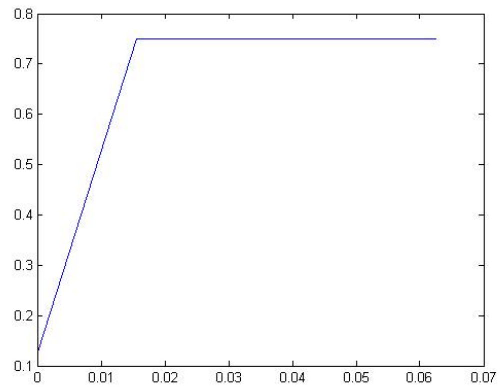

Figure 3: The quadratic approximation to the solution of the IV ODE.

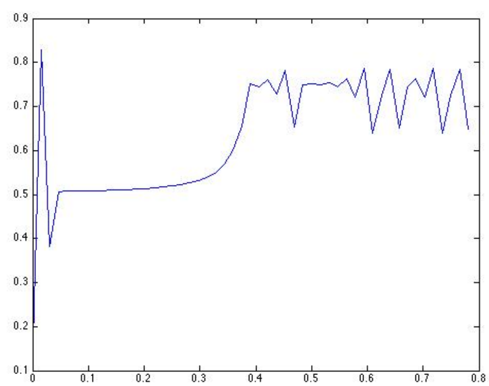

Figure 4: The cubic approximation to the solution of the IV ODE.

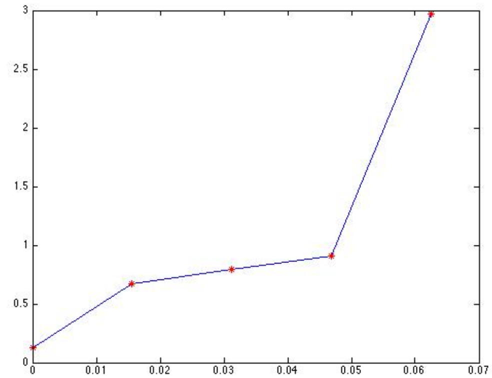

Figure 5: The quartic approximation to the solution of the IV ODE.

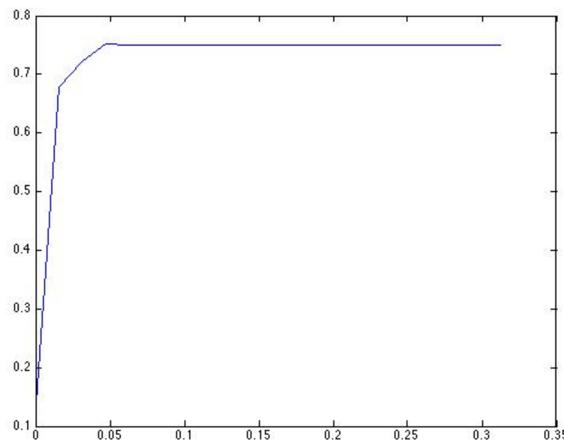

Figure 6: The 8^{th} degree approximation to the solution of the IV ODE.

This example shows that just increasing the degree of the polynomial approximation to a solution will not always give better results. However, polynomial approximations of degree greater than or equal to 8 are better approximations to the solution. It is also true that choosing a smaller h will give better approximations to the solution. The reader is now given some problems to consider in this light.

Questions, Thoughts and Projects

Question 1. Perform a similar analysis for what was done on the population dynamics example above on the IV ODE $x' = 1 + x^2;\ x(0) = 0$ that was presented in this chapter. Since you know the solution, you should ask what happens to the solution as t approaches $\frac{\pi}{2}$. Using a software package plot your approximate solutions against the exact solution to the IV ODE. From using different time steps and degrees for the approximate polynomial solutions on this IV ODE can you see that there may be the possibility of a relationship between the error in the solution and these two entities?

Question 2. Consider developing a computer program that will generate any degree polynomial approximation to the solution to the general quadratic IV ODE the user of your program desires. How can you modify this program so that it can handle time steps and give an estimate in the error between your program's approximate solution and the true solution.

Question 3. Consider the IV ODE $y' = ay^2 + by + c;\ y(t_0) = y_0$. Translate this IV ODE to $t_0 = 0$. Determine a number M so that if $x = y + M$ then x solves the quadratic IV ODE $x' = Ax^2 + C;\ x(0) = x_0$. (You should be able to determine a relationship between all the numbers.) Classify all the solutions to this IV ODE as we did in this chapter. Does knowing all the properties of the solution of one of these IV ODEs give you all the properties of the solution of the other IV ODE?