CAUCHY-KOWALEVSKI AND POLYNOMIAL ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The Cauchy-Kowalevski Theorem is the foremost result guaranteeing local existence and uniqueness for analytic quasilinear partial differential equations with Cauchy initial data. The techniques of Cauchy-Kowalevski may also be applied to initial value ordinary differential equations. These techniques, when applied in the polynomial ordinary differential equation setting, lead one naturally to a method in which coefficients are easily computed in a recursive manner, and an explicit majorization which admits a clear *a priori* error bound. The error bound depends only on immediately observable quantities of the system; coefficients, initial conditions, and polynomial degree. The numerous benefits of the polynomial system are noted for a specific example.

1. Introduction

The Cauchy-Kowalevski Theorem is the main local existence and uniqueness theorem for analytic quasilinear partial differential equations (PDE) with Cauchy initial data. Cauchy developed a proof in a restricted setting by 1842 [2], and in 1875 Kowalevski presented the full result [10]; existence

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of a unique solution to the general quasilinear system of partial differential equations given initial conditions prescribed on some non-characteristic curve. In [7], a proof in the fully nonlinear setting is presented. The Cauchy-Kowalevski argument is based on the construction of a power series solution, in which the coefficients of the series expansion are reconstructed recursively, and the method of majorants applied to verify that this solution converges locally. Convergence is demonstrated by comparison with the analytic solution of an associated PDE.

Although the Picard-Lindelöf Theorem is the fundamental local existence argument for a large class of initial value ordinary differential equations (IVODE), in 1835 Cauchy demonstrated existence and uniqueness in the ODE setting, applying a majorant based argument similar to that both he and Kowalevski would later use in the PDE setting. That is, Cauchy methods can be used to show that u satisfies the real analytic ODE $d_t u(t) = f(u(t))$, where $u(0) = u_0$ using a constructive approach, provided $f(u)$ is locally analytic near u_0. A nice treatment may be found in [4].

Given that the power series solution is directly accessible via the Cauchy-Kowalevski construction but that the method is rarely applied suggests practical difficulties. In fact, the coefficients can be tedious to construct as typically posed, as is a key constant in the comparison solution. In this paper, we demonstrate that a subtle recasting of the ODE system meliorates these difficulties: *the coefficients of the analytic solution become remarkably easy to recover, and a computable choice of the key constant leads to an a priori* *error bound.* To make these ideas clear, we consider the quasilinear IVODE

$$d_t u(t) = f(u(t)) := \frac{1}{u} \exp(-16\, u^2), \quad \text{with } u(0) = 1. \tag{1}$$

We first consider (1) using the methods of Cauchy, and identify steps in which the construction of solution becomes tedious. We then recast the problem as a polynomial system, as might be done when using Taylor series based automatic differentiation, and apply the same methods. It will be clear the computations necessary to generate the series solution are basic, and that a simple majorization which depends only on initial conditions and the constant coefficients of the polynomial system leads to an error bound. Although not demonstrated here, the method applied is quite general. See [6, 14, 16] for practical examples.

2. Cauchy solution: the classic setting

We begin with the precarious assumption that a locally analytic solution $u(t)$ to (1) exists, and repeatedly differentiate the equation, using the fact that $f(u)$ is analytic in u near the initial condition.

$$\begin{aligned}
d_t^2 u(t) &= d_u f(u) d_t u \\
&= -\frac{\mathrm{e}^{-32\,u^2}\left(32\,u^2+1\right)}{u^3} \\
d_t^3 u(t) &= d_u^2 f(u)[d_t u]^2 + d_u f(u) d_t^2 u \\
&= \frac{\mathrm{e}^{-48\,u^2}\left(2048\,u^4+96\,u^2+3\right)}{u^5} \\
d_t^4 u(t) &= d_u^3 f(u)[d_t u]^3 + 3 d_u^2 f(u) d_t^2 u d_t u + d_u f(u) d_t^3 u \\
&= -\frac{\mathrm{e}^{-64\,u^2}\left(196608\,u^6+11264\,u^4+576\,u^2+15\right)}{u^7}
\end{aligned}$$

$$\text{and} \qquad d_t^n u(t) = p_n(f(u), d_u f(u), d_u^2 f(u), \ldots, d_u^{n-1} f(u)), \tag{2}$$

where $p_n(\cdot)$ denotes a polynomial in n variables (here taken from the set of derivatives of f with respect to u of order less than n, *i.e.* $\{d_u^{k-1} f\}, k = 1, \ldots, n$, and having postive integer coefficients). By this process, all coefficients of the power series representation of $u(t)$ may be built;

$$u(t) = \sum_{k=0}^{\infty} \frac{1}{k!} d_t^k u(0)\, t^k. \tag{3}$$

Note that the form of the polynomial p_n in expression (2) allows the coefficients of the power series to be recovered recursively, although the complexity of calculation may (and usually does) grow exponentially.

By its very construction, this power series (3) yields a unique classical solution to the IVODE if it can be shown to converge. Cauchy demonstrated convergence by comparison with a related analytic IVODE, whose individual coefficients majorize (absolutely bound) those of (3). We briefly illustrate the argument. We begin with the assumption of the theorem that $f(u)$ is analytic in some interval of radius $R \in \mathbb{R}$ about $u = 1$. Then for any positive $r < R$, there exists

$$C_\infty := \max_k \{|C_k|\} < \infty, \quad \text{where} \quad C_n = \frac{1}{n!} d_u^n f(1) r^n,$$

which provides the bound

$$\max_k \left| \frac{1}{k!} d_u^k f(1) \right| \leq C_\infty r^{-k}$$

on the Taylor coefficients of $f(u)$ about $u(0) = 1$. Next we define g via the geometric series

$$g(v) := \sum_{k=0}^{\infty} C_\infty r^{-k} (v-1)^k = C_\infty \frac{r}{r-(v-1)} \quad \text{when} \quad |v-1| < r,$$

and the comparison IVODE

$$d_t v(t) = g(v(t)) \text{ with } v(0) = 1. \tag{4}$$

The form of equation (4) is motivated by the observation that the polynomial p_n generated in this case is identical in form to that of (2), allowing a direct comparison of coefficients of $u(t)$ with those of $v(t)$. Also, $g(v)$ majorizes $f(u)$ near 1 and allows (4) an analytic solution $v(t)$ near 0. When $|v-1| < r$,

$$|d_u^n f(1)| = n! \left| \frac{1}{n!} d_u^n f(1) \right| \leq n! C_\infty r^{-n} = d_v^n g(1)$$

for all n. Noting that the structure of the polynomial in (2) is identical in (1) and (4), it follows that

$$\begin{aligned}|d_t^n u(0)| &= |p_n(f(1), \ldots, d_u^{n-1} f(1))| \\ &\leq p_n(|f(1)|, \ldots, |d_u^{n-1} f(1)|) \\ &\leq p_n(g(1), \ldots, d_u^{n-1} g(1)) \\ &= d_t^n v(0),\end{aligned}$$

demonstrating that $u(t)$ is majorized by $v(t)$ in a neighborhood of $t = 0$. It follows immediately that

$$|u(t)| = \left| \sum_{k=0}^{\infty} \frac{1}{k!} d_t^k u(0)\, t^k \right| \leq \sum_{k=0}^{\infty} \frac{1}{k!} d_t^k v(0)\, |t|^k \leq v(|t|).$$

The existence of an analytic solution of (4) with radius of convergence $|t| < \frac{r}{2C_\infty}$, given by

$$v(t) = 1 + r - r\sqrt{1 - 2C_\infty t/r}, \tag{5}$$

confirms that $u(t)$ must also be locally analytic about $t = 0$.

This argument relies on C_∞, a constant which in practice is often difficult to ascertain. In our example, with $r = 1$, we have

$$\begin{aligned}C_\infty &= \max_k \{C_0, C_1, C_2, C_3, \ldots\} \\ &= \max_k \left\{ \mathrm{e}^{-16}, 33\, \mathrm{e}^{-16}, 529\, \mathrm{e}^{-16}, (16435/3)\, \mathrm{e}^{-16}, \ldots \right\},\end{aligned}$$

and it not immediately clear where the maximum might occur. An explict computation of the C_k terms, plotted in figure (1), suggests that the maximum occurs near $k = 29$, and one can easily imagine how involved the expression $d_u^{(29)} f$ has become.

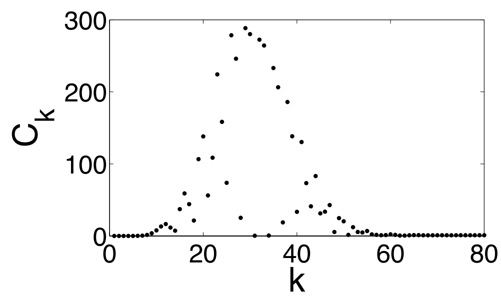

Figure 1: C_k coefficient list

3. Cauchy solution: the polynomial setting

We now apply similar techniques to an equivalent polynomial system. Recall the original problem;

$$d_t u(t) = \frac{1}{u} \exp(-16u^2), \quad \text{with } u(0) = 1.$$

Now consider the introduction of the auxiliary variables:

$$x(t) := \frac{1}{u} \exp(-16u^2) \quad \text{and} \quad y(t) := \frac{1}{u}$$

as might be introduced using the methods of [8, 15, 9, 1] when solving via automatic differentiation, or as suggested by examples treated in [13, 12, 16].

We now generate the polynomial system

$$\begin{aligned}
d_t u &= x && u(0) = 1 \\
d_t x &= (-32xu - xy)\, d_t u = -32x^2u - x^2y && x(0) = \exp(-16) \\
d_t y &= -\frac{1}{u^2} d_t u = -y^2 x && y(0) = 1.
\end{aligned}$$

The first equation is our original ODE; the additional equations serve a purely computational purpose.

As earlier, we assume the existence of an analytic solution u. We continue by assuming a formal power series for x and y, which can be shown (along with u) to be convergent via a majorant argument. Now,

$$u(t) = \sum_{k=0}^{\infty} u_k t^k, \quad x(t) = \sum_{k=0}^{\infty} x_k t^k, \quad \text{and} \quad y(t) = \sum_{k=0}^{\infty} y_k t^k.$$

The constant on which the previous argument relies is C_∞, which is difficult in general to construct. The constants related to the polynomial argument are easy to construct. In this new setting, consider the companion problem

$$d_t z = \mathcal{C} z^m \qquad z(0) = c. \tag{6}$$

Then (6) has the analytic solution

$$z\,(t) = \left(\mathcal{C}t - \mathcal{C}tm + c^{1-m}\right)^{-(m-1)^{-1}}.$$

If $\mathcal{C} = 33, m = 3$ and $c = 1$, we claim that $z(t)$ majorizes $u(t), x(t)$ and $y(t)$. These parameters arise naturally when considering the majorization; $\mathcal{C}$ from the largest row sum of the absolute value of coefficients in the system, m

from the largest degree of the polynomial system, and c from the largest of the absolute value of the initial conditions and 1. See [18] for a more detailed explanation. As a brief exercise, we demonstrate this by applying an inductive argument to verify that the coefficients of the power series representation of $z(t) = \sum_{k=0}^{\infty} z_k t^k$ bound those of $x(t)$. Clearly $z_0 \geq |x_0|$, since $c \geq |\exp(-16)|$, the initial condition. Obviously,

$$z_1 = 33z_0^3 \geq |-32x_0^2u_0 - x_0^2y_0| = |x_1|.$$

Assuming $z_k > \{|u_k|, |x_k|, |y_k|\}$ for $k = 0, \dots, n$, it follows that

$$\begin{aligned} z_{n+1} &= \frac{1}{n+1} \cdot 33 \sum_{k=0}^{n} \left(\sum_{i=0}^{k} z_i z_{k-i} \right) z_{n-k} \\ &\geq \frac{1}{n+1} \cdot \left| -32 \sum_{k=0}^{n} \left(\sum_{i=0}^{k} x_i x_{k-i} \right) u_{n-k} - \sum_{k=0}^{n} \left(\sum_{i=0}^{k} x_i x_{k-i} \right) y_{n-k} \right| \qquad (7) \\ &= |x_{n+1}| \end{aligned}$$

where a Cauchy product has been applied twice. An important (and obvious) observation used in (7) is that

$$x_{n+1} = \frac{1}{n+1} \cdot \left[-32 \sum_{k=0}^{n} \left(\sum_{i=0}^{k} x_i x_{k-i} \right) u_{n-k} - \sum_{k=0}^{n} \left(\sum_{i=0}^{k} x_i x_{k-i} \right) y_{n-k} \right],$$

which can easily be implemented to construct the coefficient x_{n+1} using only coefficients of order n or less. The software tools `ATOMFT` and `Taylor` are two such packages that exploit this recursive feature [3, 9]. The polynomial used to construct coefficients in the classic setting, p_n, has now been replaced by an algebraic expression whose complexity is only $\mathcal{O}(n^3)$. (In fact, augmenting the system allows reduction to $\mathcal{O}(n^2)$ [17].) Since $z(t)$ converges on some open interval containing $t = 0$ and majorizes $x(t)$ for $|t| < 1$, $x(t)$ must also converge on the intersection of these intervals. The demonstration is now complete; an explicit verification that $x(t)$ converges via a term-by-term comparison with the convergent series representation of $z(t)$. It is easy to see that a similar argument may be used for $u(t)$ and $y(t)$.

In addition to a simple coefficient recursion and explicit majorization, the polynomial comparison solution gives rise to an easily computable local *a priori* error bound. To accomplish this, the comparison solution $z(t)$ is bounded by $w(t)$, a function with a geometric series representation. We begin with the recurrence relation for the coefficients of z,

$$z_{n+1} = \frac{(1+(m-1)n)c^{m-1}\mathcal{C}}{n+1} z_n \qquad z_0 = c, \text{ for } n \geq 1. \tag{8}$$

For $m \geq 2$,

$$\frac{(1+(m-1)n)c^{m-1}\mathcal{C}}{n+1} \leq (m-1)c^{m-1}\mathcal{C} := \mathcal{C}_\infty. \tag{9}$$

Combining (8) and (9) yields $z_{n+1} \leq \mathcal{C}_\infty z_n$. If

$$w_{n+1} = \mathcal{C}_\infty w_n, \qquad \text{with } w_0 = c, \tag{10}$$

then the coefficients of w majorize those of z (and therefore u), and $w(t)$ majorizes $z(t)$ (and $u(t)$). The recurrence relation (10) leads directly to the geometric series,

$$w(t) = \frac{c}{1-\mathcal{C}_\infty t} = c\sum_{k=0}(\mathcal{C}_\infty t)^k, \quad \text{when} \quad |t| < \frac{1}{\mathcal{C}_\infty}.$$

The function w may be interpreted as a solution to the IVODE

$$d_t w(t) = \mathcal{C}_\infty w, \qquad w(0) = c \tag{11}$$

where $\mathcal{C}_\infty$ bounds the coefficient growth of terms of z, playing much the same role as C_∞. Here, however, $\mathcal{C}_\infty$ is trivial to compute from (9).

Finally, a simple bound on the remainder term $\mathcal{R}_n$, given by

$$\begin{aligned}\mathcal{R}_n(t) := \left| u(t) - \sum_{k=0}^{n} u_k t^k \right| &\le c \sum_{k=n+1}^{\infty} \mathcal{C}_\infty |t|^k \\ &\le c|\mathcal{C}_\infty t|^{n+1} \frac{1}{1 - |\mathcal{C}_\infty t|},\end{aligned} \tag{12}$$

provides a concise and computable error bound. For (1),

$$\mathcal{R}_n(t) \le \frac{|66t|^{n+1}}{1 - |66t|}.$$

For a detailed discussion, and an example for which this bound is tight, see [18]. See [11] for a detailed discussion of Interval Analysis, an alternative approach.

4. Conclusion

We have demonstrated that recasting the original ODE as a polynomial system has several surprising benefits. The techniques of Cauchy-Kowalevski, when applied to a polynomial system, lead one naturally to a method in which; *i)* coefficients are easily computed in a recursive manner, *i.e.* $u_{n+1}, x_{n+1},$ and y_{n+1} only depend on products and sums of $\{u_k, x_k, y_k\}_{k=1..n}$, *ii)* the majorization is explicit, and *iii)* there is a clear *a priori* error bound.

The error bound depends only on immediately observable quantities of the recast system; coefficient sums, initial conditions, and degree.

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