1.

$$x = 1 + t \ ; \ y = \frac{1}{x} = \frac{1}{1+t} \ ; \ y(0) = 1$$

If $g = \frac{1}{f}$ then $gf = 1$ and if $f = \sum_{k=0}^{\infty} f_k t^k$ then

$$g_0 = \frac{1}{f_0} \ ; \ g_n = -\frac{1}{f_0} \sum_{j=0}^{n-1} f_{n-j} g_j$$

$$y' = -y^2 \ ; \ y(0) = 1$$

$$y' = \sum_{n=0}^{\infty} (n+1) b_{n+1} t^n = -(\sum_{n=0}^{\infty} b_n t^n)^2 = -b_0^2 - \sum_{n=1}^{\infty} [\sum_{j=0}^{n} b_{n-j} b_j] t^i$$

2.

$$x' = 1 + x^2 \ ; \ x(0) = 0$$

$$x = \tan t$$

$$x = \sum a_i t^i$$

$$x' = \sum_{i=0}^{\infty}(i+1)a_{i+1}t^i = 1 + (\sum_{i=0}^{\infty} a_i t^i)^2 = 1 + a_0^2 + \sum_{i=1}^{\infty}[\sum_{j=0}^{i} a_j a_{i-j}]t^i$$

$$x = \frac{\sin t}{\cos t}$$

If $h(t) = \frac{f(t)}{g(t)}$ then since $h(t)g(t) = f(t)$; $h_0 = \frac{f_0}{g_0}$ and

$$h_i = \frac{1}{g_0}(f_i - \sum_{j=0}^{i-1} g_j h_{i-j}) \ ; \ i = 1, 2, 3, ...$$

PROJECTIVELY POLYNOMIAL

Let $f : \mathbb{R} \to \mathbb{R}$ with 0 in the domain of f. The following are equivalent definitions of f being projectively polynomial.

(1) There are natural numbers n, k, a polynomial $P : \mathbb{R}^n \to \mathbb{R}^n$ of degree k and a vector $A \in \mathbb{R}^n$ so that if

$$y' = P(y) \ ; \ y(0) = A$$

then $f = y_1$.

We write $f \in \mathbb{P}_{n,k}$. We let $\mathbb{P}_n = \cup_k \mathbb{P}_{n,k}$ and $\mathbb{P} = \cup_n \mathbb{P}_n$. We can also write $f \in \mathbb{P}_n$ or $f \in \mathbb{P}$.

(2) There exists a polynomial $Q : \mathbb{R}^{n+1} \to \mathbb{R}$ so that $Q(f, f', ..., f^{(n)}) = 0$.

(3) There exists a natural number N and real numbers $a_1, ..., a_N$; and $b_{1,1}, ..., b_{1,N}, ..., b_{N,1}, ..., b_{N,N}$; and $c_{1,1,1}, ..., c_{1,1,N}, c_{1,2,2}, ..., c_{1,2,N}, ..., c_{1,N,N}, ..., c_{2,1,1}, ..., c_{2,1,N}, c_{2,2,2}, ..., c_{2,2,N}, ..., c_{2,N,N}, ...,$ $...c_{N,1,1}, ..., c_{n,1,N}, c_{N,2,2}, ..., c_{N,2,N}, ..., c_{N,N,N}$; and $B_1, ..., B_N$; together with functions $x_1, ..., x_N$; so that if for $j = 1, ..., N$

$$x_j' = a_j + \sum_{i=1}^{N} b_{j,i} x_i + \sum_{i=1}^{N} \sum_{k=i}^{N} c_{j,i,k} x_i x_k \ ; \ x_j(0) = B_j$$

and $f = x_1 = \sum_{k=0}^{\infty} \alpha_{1,k} t^k$.

(4) There exists a natural number N and real numbers $a_1, ..., a_N$; and $b_{1,1}, ..., b_{1,N}, ..., b_{N,1}, ..., b_{N,N}$; and $c_{1,1,1}, ..., c_{1,1,N}, c_{1,2,2}, ..., c_{1,2,N}, ..., c_{1,N,N}, ..., c_{2,1,1}, ..., c_{2,1,N}, c_{2,2,2}, ..., c_{2,2,N}, ..., c_{2,N,N}, ...,$ $...c_{N,1,1}, ..., c_{n,1,N}, c_{N,2,2}, ..., c_{N,2,N}, ..., c_{N,N,N}$; together with sequences $\alpha_1, ..., \alpha_N$; so that $\alpha_{1,0} = f(0) = A_1$ and

$$\alpha_{j,1} = a_j + \sum_{i=1}^{N} b_{j,i} \alpha_{i,0} + \sum_{m=1}^{N} \left(\sum_{i=m}^{N} c_{j,m,i} \alpha_{m,0} \alpha_{i,0} \right)$$

$$\alpha_{j,k} = \frac{1}{k} \left(\sum_{i=1}^{N} b_{j,i} \alpha_{i,k-1} + \sum_{m=1}^{N} \left(\sum_{i=m}^{N} (c_{j,m,i} \sum_{l=0}^{k-1} \alpha_{m,j} \alpha_{i,k-l-1}) \right) \right)$$

for $j = 1, ..., N$ and $f = \sum_{k=0}^{\infty} \alpha_{1,k} t^k$.

(5) There exists a natural number N and real numbers $a_1, ..., a_N$; and $b_{1,1}, ..., b_{1,N}, ..., b_{N,1}, ..., b_{N,N}$; and $c_{1,1}, ..., c_{1,N}, c_{2,1}, ..., c_{2,N}, ..., c_{N,1}, ..., c_{N,N}$; together with sequences $\alpha_1, ..., \alpha_N$; so that $\alpha_{1,0} = f(0) = A_1$ and

$$y_j' = a_j + \sum_{m=1}^{n} b_{j,m} y_m + \sum_{m=1}^{n} c_{j,m} {y_m}^2$$

(6) There exists a natural number N and real numbers $a_1, ..., a_N$; and $b_{1,1}, ..., b_{1,N}, ..., b_{N,1}, ..., b_{N,N}$; and $c_{1,1}, ..., c_{1,N}, c_{2,1}, ..., c_{2,N}, ..., c_{N,1}, ..., c_{N,N}$; together with sequences $\alpha_1, ..., \alpha_N$; so that $\alpha_{1,0} = f(0) = A_1$ and

$$\alpha_{N,k} = \frac{\sum_{j=1}^{J} b_{N,j} \alpha_{j,k-1} + \sum_{j=1}^{J} \left(c_{N,j} \sum_{m=0}^{k-1} \alpha_{j,m} \alpha_{j,k-m-1} \right)}{k}$$

$$f(t) = \sum_{i=0}^{\infty} f_i t^i = [f_0, f_1, f_2, ...].$$

Note that $t[g_0, g_1, g_2, ...] = [0, g_0, g_1, g_2, ...]$, $t^2[g_0, g_1, g_2, ...] = [0, 0, g_0, g_1, g_2, ...]$, etc. That is, t^k times a power series g is a power series for g with the first k coefficients being 0 and the remaining coefficients the same as that of g. Therefore,

$$g(t)h(t) = [g_0, g_1, g_2, ...][h_0, h_1, h_2, ...] = g_0[h_0, h_1, h_2, ...] + g_1[0, h_0, h_1, h_2, ...] + g_2[0, 0, h_0, h_1, h_2, ...] + ...$$

Therefore, the coefficients of the power series $g(t)h(t)$ is given by the product of the circulant matrix

$$\begin{pmatrix} g_0 & 0 & 0 & . & . & . & 0 \\ g_1 & g_0 & 0 & . & . & . & 0 \\ g_2 & g_1 & g_0 & 0 & . & . & 0 \\ g_3 & g_2 & g_1 & g_0 & 0 & ... & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix} \text{ and the vector } \begin{pmatrix} h_0 \\ h_1 \\ h_2 \\ h_3 \\ . \\ . \end{pmatrix},$$