Polynomial ODEs - Examples, Solutions, Properties

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Abstract. Let P be a polynomial from $\mathbb{R}^n \to \mathbb{R}^n$ and $D \in \mathbb{R}^n$. I consider some properties of the class of ODEs $Y' = P(Y)$; $Y(0) = D$ and their solutions. The solution space to these ODEs form a proper subspace of the analytic functions. I will present examples highlighting the concept and practicality of polynomial ODEs including Newton's N-Body problem and root finding methods.

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Example

$$x' = 1 + x^2 \ ; \ x(0) = 0.$$

$$x = \sum a_i t^i$$

$$x' = \sum_{i=0}^{\infty} (i+1) a_{i+1} t^i = 1 + (\sum_{i=0}^{\infty} a_i t^i)^2 = 1 + a_0^2 + \sum_{i=1}^{\infty} [\sum_{j=0}^{i} a_j a_{i-j}] t^i$$

$$a_0 = x(0) \ ; \ a_1 = 1 + a_0^2 \ ; \ a_{i+1} = \frac{1}{i+1} \sum_{j=0}^{i} a_i a_{j-i} \ ; \ i = 1, 2, 3, ...$$

$$f(t) = \sum_{i=0}^{\infty} f_i t^i = [f_0, f_1, f_2, ...]$$

$$g(t)h(t) = [g_0, g_1, g_2, ...][h_0, h_1, h_2, ...] = g_0[h_0, h_1, h_2, ...] + g_1[0, h_0, h_1, h_2, ...] + g_2[0, 0, h_0, h_1, h_2, ...] + ...$$

$$\begin{pmatrix} h_0 & 0 & 0 & . & . & . & 0 \\ h_1 & h_0 & 0 & . & . & . & 0 \\ h_2 & h_1 & h_0 & 0 & . & . & 0 \\ h_3 & h_2 & h_1 & h_0 & 0 & ... & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \\ . \\ . \end{pmatrix}$$

$$h(t) = \frac{f(t)}{g(t)} \qquad h(t)g(t) = f(t)$$

$$h_0 = \frac{f_0}{g_0}$$

$$h_i = \frac{1}{g_0}(f_i - \sum_{j=0}^{i-1} g_j h_{i-j}) \; ; \; i = 1, 2, 3, \ldots$$

$$p = \frac{1}{g}$$

$$p_0 = \frac{1}{g_0}\ ;\ p_1 = \frac{1}{g_0}(1 - g_1 p_0)\ ;\ p_i = -\frac{1}{g_0}\sum_{j=0}^{i-1} g_j p_{i-j}, \qquad i = 2, 3, 4, ...$$

$$(pg)' = p'g + pg' = 0.$$

$$p' = -\frac{1}{g^2}g' = -p^2 g'$$

$$p' = -wg'\ ;\ w' = 2pp'$$

$$(i+1)p_{i+1} = \sum_{j=0}^{i}(j+1)g_{j+1}w_{i-j}\ ; (i+1)w_{i+1} = 2\sum_{j=0}^{i}(j+1)p_{j+1}p_{i-j}$$

$$x(t) = \frac{t}{e^t - 1}$$

$$y(t) = \frac{1}{x(t)} = \frac{e^t - 1}{t}$$

$$y(t) = [1, \frac{1}{2}, \frac{1}{3!}, \frac{1}{4!}, \ldots]$$

$$\frac{f(t) - f_0}{t} = [f_1, f_2, f_3, \ldots]$$

$$x(t) = \frac{t}{e^t-1} = [1, -1/2, 1/12, 0, -\frac{1}{720}, 0, \frac{1}{30240}, 0, -\frac{1}{1209600}, 0, \frac{1}{47900160}, 0, -\frac{691}{1307674368000}, 0, \frac{1}{74724249600}, 0, -\frac{3617}{10670622842880000}, 0, \frac{43867}{5109094217170944000}, 0, -\frac{174611}{802857662698291200000}, 0, \ldots].$$

PROJECTIVELY POLYNOMIAL

Definition. The function $f : \mathbb{R} \to \mathbb{R}$ is *projectively polynomial* if there are natural numbers n, k, a polynomial $P : \mathbb{R}^n \to \mathbb{R}^n$ of degree k and a vector $A \in \mathbb{R}^n$ so that if

$$y' = P(y) \ ; \ y(0) = A$$

then $f = y_1$. Note that this implies f is analytic on a neighborhood of 0 (i.e. $f \in \mathbb{A}$) and $f(0) = A_1$. We write $f \in \mathbb{P}_{n,k}$. We let $\mathbb{P}_n = \cup_k \mathbb{P}_{n,k}$ and $\mathbb{P} = \cup_n \mathbb{P}_n$. Parker and Sochacki have shown that many functions are projectively polynomial and that many ODEs can be made polynomial through auxiliary variables (Parker & Sochacki, 1996).

The following have been shown to be equivalent statements for f being projectively polynomial (Carothers et al., 2005).

(1) There exists a polynomial $Q : \mathbb{R}^{n+1} \to \mathbb{R}$ so that $Q(f, f', ..., f^{(n)}) = 0$.

(2) There exists a natural number N and real numbers $a_1, ..., a_N$; and $b_{1,1}, ..., b_{1,N}, ..., b_{N,1}, ..., b_{N,N}$; and $c_{1,1,1}, ..., c_{1,1,N}, c_{1,2,2}, ..., c_{1,2,N}, ..., c_{1,N,N}, ..., c_{2,1,1}, ..., c_{2,1,N}, c_{2,2,2}, ..., c_{2,2,N}, ..., c_{2,N,N}, ...,$ $...c_{N,1,1}, ..., c_{n,1,N}, c_{N,2,2}, ..., c_{N,2,N}, ..., c_{N,N,N}$; and $B_1, ..., B_N$; together with functions $x_1, ..., x_N$; so that if for $j = 1, ..., N$

$$x_j' = a_j + \sum_{i=1}^{N} b_{j,i} x_i + \sum_{i=1}^{N} \sum_{k=i}^{N} c_{j,i,k} x_i x_k \; ; \; x_j(0) = B_j$$

then $f = x_1 = \sum_{k=0}^{\infty} \alpha_{1,k} t^k$.

(3) There exists a natural number N and real numbers $a_1, ..., a_N$; and $b_{1,1}, ..., b_{1,N}, ..., b_{N,1}, ..., b_{N,N}$; and $c_{1,1,1}, ..., c_{1,1,N}, c_{1,2,2}, ..., c_{1,2,N}, ..., c_{1,N,N}, ..., c_{2,1,1}, ..., c_{2,1,N}, c_{2,2,2}, ..., c_{2,2,N}, ..., c_{2,N,N}, ...,$ $...c_{N,1,1}, ..., c_{n,1,N}, c_{N,2,2}, ..., c_{N,2,N}, ..., c_{N,N,N}$; together with sequences $\alpha_1, ..., \alpha_N$; so that $\alpha_{1,0} = f(0) = A_1$ and

$$\alpha_{j,1} = a_j + \sum_{i=1}^{N} b_{j,i} \alpha_{i,0} + \sum_{m=1}^{N} \left(\sum_{i=m}^{N} c_{j,m,i} \alpha_{m,0} \alpha_{i,0} \right)$$

$$\alpha_{j,k} = \frac{1}{k} \left(\sum_{i=1}^{N} b_{j,i} \alpha_{i,k-1} + \sum_{m=1}^{N} \left(\sum_{i=m}^{N} (c_{j,m,i} \sum_{l=0}^{k-1} \alpha_{m,j} \alpha_{i,k-l-1}) \right) \right)$$

for $j = 1, ..., N$ and $f = \sum_{k=0}^{\infty} \alpha_{1,k} t^k$.

Consider

$$x' = -x^r \ ; \ x(0) = A$$

$$x_2 = x^r$$

$$x' = -x_2 \ ; \ x(0) = A$$
$$x_2' = rx^{r-1} \ ; \ x_2(0) = A^r$$

$$x_3 = x^{-1}$$

$$x' = -x_2 \ ; \ x(0) = A$$
$$x_2' = -rx_2^2x_3 \ ; \ x_2(0) = A^r$$
$$x_3' = x_2x_3^2 \ ; \ x_3(0) = \frac{1}{A}.$$

$$x_4 = x_2x_3$$

$$x' = -x_2 \ ; \ x(0) = A$$
$$x_2' = -rx_2x_4 \ ; \ x_2(0) = A^r$$
$$x_3' = x_3x_4 \ ; \ x_3(0) = \frac{1}{A}$$
$$x_4' = (1-r)x_4^2 \ ; \ x_4(0) = A^{r-1}.$$

$$x' = -xx_2 \ ; \ x(0) = A$$
$$x_2' = (1-r)x_2^2 \ ; \ x_4(0) = A^{r-1}.$$

$$x'' = -x'x_2 - xx_2' = \frac{(x')^2}{x} + (r-1)xx_2^2 = \frac{(x')^2}{x} + (r-1)\frac{(x')^2}{x}$$
$$xx'' - r(x')^2 = 0.$$

$$x' = \sin x;\ \ x(0) = \alpha$$

$$x_2 = \sin x \text{ and } x_3 = \cos x$$

$$\begin{aligned}
x' &= x_2;\ \ x(0) = \alpha \\
x_2' &= x_2 x_3;\ \ x_2(0) = \sin(\alpha) \\
x_3' &= -x_2^2;\ \ x_3(0) = \cos(\alpha).
\end{aligned}$$

$$x_3' = x_3^2 - 1$$

$$x_3 = \frac{1 - e^{2t+2B}}{1 + e^{2t+2B}} \qquad x_2 = \frac{4e^{2t+2B}}{(1 + e^{2t+2B})^2}. \qquad x = 2 \arctan e^{t+B}$$

$$\begin{aligned}
&x'' = x_2' = x_2 x_3 \\
&x''' = x_2' x_3 + x_2 x_3' = x_2 x_3^2 - x_2^3 = x_2(1 - x_2^2) - x_2^3 \\
&x''' - x' + 2(x')^3 = 0.
\end{aligned}$$

Single Pendulum

$$\ddot{\theta} = -\frac{g}{r}\cos\theta$$

$$y_1 = \theta \ ; \ y_2 = \theta \ ; \ y_3 = \sin\theta \ ; \ y_4 = \cos\theta$$

$$y_1' = y_2 \ ; \ y_1(0) = \theta(0)$$

$$y_2' = -\tfrac{g}{r} y_4 \ ; \ y_2(0) = \dot{\theta}(0)$$

$$y_3' = y_2 y_4 \ ; \ y_3(0) = \sin(\theta(0))$$

$$y_4' = -y_2 y_3 \ ; \ y_4(0) = \cos(\theta(0))$$

$$y_1 = \textstyle\sum_{i=0}^{\infty} a_i t^i \ ; \ y_2 = \sum_{i=0}^{\infty} b_i t^i \ ; \ y_3 = \sum_{i=0}^{\infty} c_i t^i \ ; \ y_4 = \sum_{i=0}^{\infty} d_i t^i$$

$$a_{i+1} = \tfrac{b_i}{i+1} \ ; \ b_{i+1} = -\tfrac{g}{r} d_i i + 1 \ ; \ c_{i+1} = \tfrac{(\sum_{j=0}^{i} b_i d_{j-i})}{i+1} \ ; \ d_{i+1} = \tfrac{[-(\sum_{j=0}^{i} b_i c_{j-i})]}{i+1} \ ; \ i = 0, 1, 2, \ldots$$

$$y_2'' = -\frac{g}{r} y_4' = \frac{g}{r} y_2 y_3 \qquad \left(\frac{y_2''}{y_2}\right)' = \frac{g}{r} y_3' = \frac{g}{r} y_2 y_4 = -y_2 y_2'$$

$$\frac{1}{2}(y_2^2) + \left(\frac{y_2''}{y_2}\right) = c \qquad y_2'' + \frac{1}{2}(y_2^3) - c y_2 = 0$$

Newton's N body ODEs for the positions (x_i, y_i, z_i) of $i = 1, .., N$ bodies

$$x_i''(t) = \sum_{j \neq i} \frac{m_j(x_j - x_i)}{r_{i,j}^{\frac{3}{2}}} \; ; \; y_i''(t) = \sum_{j \neq i} \frac{m_j(y_j - y_i)}{r_{i,j}^{\frac{3}{2}}} \; ; \; z_i''(t) = \sum_{j \neq i} \frac{m_j(z_j - z_i)}{r_{i,j}^{\frac{3}{2}}}$$

$$r_{i,j} = [(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2], j = 1, ..., N$$

$$s_{i,j} = r_{i,j}^{-\frac{1}{2}}$$

$$x_i' = u_i \; ; \; y_i' = v_i \; ; \; z_i' = w_i$$

$$u_i' = \sum_{j \neq i} m_j(x_j - x_i) s_{i,j}^3 \; ; \; v_i' = \sum_{j \neq i} m_j(y_j - y_i) s_{i,j}^3 \; ; \; w_i' = \sum_{j \neq i} m_j(z_j - z_i) s_{i,j}^3$$

$$s_{i,j}' = -\frac{1}{2} s_{i,j}^3 [2(x_i - x_j)(u_i - u_j) + 2(y_i - y_j)(v_i - v_j) + 2(z_i - z_j)(w_i - w_j)], \qquad i, j = 1, .., N.$$

$$f(t) = \sum_{i=0}^{n+2} a_i t^i = a_0 + a_1 t + ... + a_{n+2} t^{n+2} \qquad f(f^{-1}(t)) = t$$

If we let $z = f^{-1}(t)$ we have $f(z) = t, f'(z)z' = 1$, and $z' = \frac{1}{f'(z)} = [f'(z)]^{-1} = y$.

$$y' = -y^2 f''(z)z' = -xf''(z)z' = -xp_n z' = -xyp_n \ ; \ (x = y^2) \ ; \ (p_n = f''(z))$$

$$x' = 2yy'$$

$$p'_n = f'''(z)z' = p_{n-1}y$$

$$p'_{n-1} = f^{(iv)}(z)z' = p_{n-2}y$$

$$p'_1 = f^{(n+2)}(z)z' = (n+2)!a_{n+2}y.$$

$$y = \sum_{i=0}^{K} y_i t^i \ ; \ y' = \sum_{i=0}^{K-1} (i+1) y_{i+1} t^i \ ; \ x = \sum_{i=0}^{K} x_i t^i$$

$$p_{n-k} = \sum_{i=0}^{K} p_{(n-k),i} t^i \ ; \ k = 0, ..., n-1.$$

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad x' = -\frac{f(x)}{f'(x)}$$

$$x' = -f(x)x_2 \qquad x_2' = -x_2^3 f''(x)$$

$$x' = -[Df(x)]^T f(x) = G(x)$$

$$x_{k+1} = x_k + \alpha_k \nabla f(x_k) \qquad x'(t) = \alpha(t) \nabla f(x)$$

$$x' = -\alpha(t)[Df(x)]^T f(x)$$

$$u_t + (f(u))_x + \mu^2 u_{xx} = 0; \quad u(x,0) = \alpha(x)$$

$$w = f(u)$$

$$\begin{array}{ll} u_t + w_x + \mu^2 u_{xx} = 0; & u(x,0) = \alpha(x) \\ w_t + f'(u)(w_x + \mu^2 u_{xx}) = 0; & w(x,0) = f(u(x,0)) = \beta(x) \end{array}$$

If $w = u^2/2$ and $\mu = 0$ then

$$\begin{array}{ll} u_t = -w_x \ ; & u(0,x) = \alpha(x) \\ w_t = uu_t = -uw_x \ ; & w(0,x) = \frac{\alpha(x)^2}{2} \end{array}$$

If $w_k = \sum_{n=0}^{k} b_n t^n$ then $b_0(x) = \frac{\alpha^2(x)}{2}$

$$u_{k+1} = u_k + a_{k+1}(x) t^{k+1} = u_k - \frac{b'_k}{k+1} t^{k+1}$$

$$w_{k+1} = w_k + b_{k+1} t^{k+1} = w_k - \frac{\sum_{n=0}^{k} a_n b'_{k-n}}{k+1} t^{k+1}$$