

Cauchy-Kowalevski and Polynomial ODE

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Why?

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- Cauchy-Kowalevski : fundamental existence uniqueness of IBVPDE.

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- Cauchy-Kowalevski : fundamental existence uniqueness of IBVPDE.

But Cauchy arguments can be used in ODE setting, too. If the ODE is polynomial, the reasoning leads to an easy iterative construction and a clear *a priori* error bound.

A brief history

- Cauchy (1835)

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A brief history

Cauchy and Weierstrass **independently** developed the **methods of majorants**. Idea is to bound a series by one which is known to converge. If the bounding function is analytic, then so is the function represented by the original series.

Cauchy approach to ODE

The ODE:

$$d_t u(t) = f(u(t)) := \frac{1}{u} \exp(-16 u^2), \quad \text{with } u(0) = 1.$$

What now?

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The ODE:

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What now?

$$\begin{aligned} d_t^2 u(t) &= d_u f(u) d_t u \\ &= -\frac{e^{-32 u^2} (32 u^2 + 1)}{u^3} \end{aligned}$$

$$\begin{aligned} d_t^3 u(t) &= d_u^2 f(u) [d_t u]^2 + d_u f(u) d_t^2 u \\ &= \frac{e^{-48 u^2} (2048 u^4 + 96 u^2 + 3)}{u^5} \end{aligned}$$

$$\text{and } d_t^n u(t) = p_n(f(u), d_u f(u), d_u^2 f(u), \dots, d_u^{n-1} f(u)), \quad (1)$$

Cauchy approach to ODE

Now just build a series representation of u .

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What's wrong? It looks so easy...

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$$u(t) = \sum_{k=0}^{\infty} \frac{1}{k!} d_t^k u(0) t^k. \quad (2)$$

It gets complicated really fast!

Cauchy approach to ODE

Consider even the fourth derivative...

$$d_t^4 u(t) = d_u^3 f(u)[d_t u]^3 + 3d_u^2 f(u)d_t^2 u d_t u + d_u f(u)d_t^3 u \quad (3)$$

$$= -\frac{e^{-64 u^2} (196608 u^6 + 11264 u^4 + 576 u^2 + 15)}{u^7} \quad (4)$$

In general, this is daunting.

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really, though, “we” can produce a closed form...

$$\frac{\sum_{k=0}^n \binom{n}{k} 2^{-k} G_{2,3}^{1,2} \left(16 \left| \begin{matrix} 0, 1/2 \\ 0, 1/2 + 1/2 - k, 1/2 - k \end{matrix} \right. \right) \text{poch}(-n + k, n - k)}{n!}$$

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Remember – this is just the candidate solution. We need to verify that this converges.

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Consider:

$$C_\infty := \max_k \{|C_k|\} < \infty, \quad \text{where} \quad C_n = \frac{1}{n!} d_u^n f(1) r^n,$$

which provides the bound

$$\max_k \left| \frac{1}{k!} d_u^k f(1) \right| \leq C_\infty r^{-k}$$

on the Taylor coefficients of $f(u)$ about $u(0) = 1$

Cauchy approach to ODE

Now for the comparison IODE:

$$d_t v(t) = g(v(t)) \text{ with } v(0) = 1. \quad (5)$$

where

$$g(v) := \sum_{k=0}^{\infty} C_{\infty} r^{-k} (v-1)^k = C_{\infty} \frac{r}{r - (v-1)} \quad \text{when } |v-1| < r,$$

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Same form, same universal polynomial as

$$d_t u(t) = f(u(t)) \text{ with } u(0) = 1.$$

Cauchy approach to ODE

it follows that

$$\begin{aligned} |d_t^n u(0)| &= |p_n(f(1), \dots, d_u^{n-1} f(1))| \\ &\leq p_n(|f(1)|, \dots, |d_u^{n-1} f(1)|) \\ &\leq p_n(g(1), \dots, d_u^{n-1} g(1)) \\ &= d_t^n v(0), \end{aligned}$$

demonstrating that $u(t)$ is majorized by $v(t)$ in a neighborhood of $t = 0$.

Cauchy approach to ODE

Now since

$$|u(t)| = \left| \sum_{k=0}^{\infty} \frac{1}{k!} d_t^k u(0) t^k \right| \leq \sum_{k=0}^{\infty} \frac{1}{k!} d_t^k v(0) |t|^k \leq v(|t|),$$

and

$$v(t) = 1 + r - r\sqrt{1 - 2C_{\infty}t/r}, \quad (6)$$

we know that $u(t)$ must also be locally analytic about $t = 0$.

Cauchy approach to ODE

This argument relies on C_∞ , a constant which in practice is often difficult to ascertain.

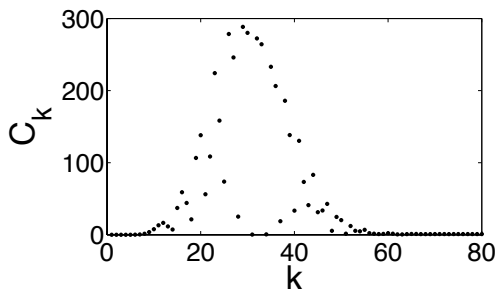


Figure: C_k coefficient list

An argument for polynomials

Recall:

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And build a system of polynomial ODE:

$$d_t u = x \qquad u(0) = 1$$

$$d_t x = (-32xu - xy) d_t u = -32x^2u - x^2y \qquad x(0) = \exp(-16)$$

$$d_t y = -\frac{1}{u^2} d_t u = -y^2 x \qquad y(0) = 1.$$

An argument for polynomials

The *new* companion IODE

$$d_t z = C z^m \quad z(0) = c, \quad (7)$$

which has analytic solution

$$z(t) = (Ct - Ctm + c^{1-m})^{-(m-1)^{-1}}.$$

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We claim that $z(t)$ majorizes $u(t)$, $x(t)$ and $y(t)$, if $C = 33$, $m = 3$ and $c = 1$.

An argument for polynomials

Consider just

$$d_t x = -32x^2 u - x^2 y \quad x(0) = \exp(-16)$$

But we choose $C = 33$, $m = 3$, and $c = 1$.

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and

$$\begin{aligned} z_{n+1} &= \frac{1}{n+1} \cdot 33 \sum_{k=0}^n \left(\sum_{i=0}^k z_i z_{k-i} \right) z_{n-k} \\ &\geq \frac{1}{n+1} \cdot \left| -32 \sum_{k=0}^n \left(\sum_{i=0}^k x_i x_{k-i} \right) u_{n-k} - \sum_{k=0}^n \left(\sum_{i=0}^k x_i x_{k-i} \right) y_{n-k} \right| \\ &= |x_{n+1}| \end{aligned} \tag{8}$$

An argument for polynomials

IMPORTANT

$$x_{n+1} = \frac{1}{n+1} \cdot \left[-32 \sum_{k=0}^n \left(\sum_{i=0}^k x_i x_{k-i} \right) u_{n-k} - \sum_{k=0}^n \left(\sum_{i=0}^k x_i x_{k-i} \right) y_{n-k} \right]$$

An error bound for polynomials

From

$$d_t z = \mathcal{C} z^m \quad z(0) = c, \quad (9)$$

we can extract the recurrence relation:

$$z_{n+1} = \frac{(1 + (m-1)n)c^{m-1}\mathcal{C}}{n+1} z_n \quad z_0 = c, \text{ for } n \geq 1. \quad (10)$$

An error bound for polynomials

For $m \geq 2$,

$$\frac{(1 + (m - 1)n)c^{m-1}\mathcal{C}}{n + 1} \leq (m - 1)c^{m-1}\mathcal{C} := \mathcal{C}_\infty. \quad (11)$$

Combining (10) and (11) yields $z_{n+1} \leq \mathcal{C}_\infty z_n$. If

$$w_{n+1} = \mathcal{C}_\infty w_n, \quad \text{with } w_0 = c, \quad (12)$$

An argument for polynomials

From this recurrence,

$$w(t) = \frac{c}{1 - \mathcal{C}_\infty t} = c \sum_{k=0}^{\infty} (\mathcal{C}_\infty t)^k, \quad \text{when } |t| < \frac{1}{\mathcal{C}_\infty}.$$

The function w may be interpreted as a solution to the IODE

$$d_t w(t) = \mathcal{C}_\infty w, \quad w(0) = c \tag{13}$$

where \mathcal{C}_∞ bounds the coefficient growth of terms of z , playing much the same role as C_∞ .

An argument for polynomials

Now for a simple bound on the remainder term \mathcal{R}_n !

$$\begin{aligned}\mathcal{R}_n(t) &:= \left| u(t) - \sum_{k=0}^n u_k t^k \right| \leq c \sum_{k=n+1}^{\infty} C_{\infty} |t|^k \\ &\leq c |C_{\infty} t|^{n+1} \frac{1}{1 - |C_{\infty} t|}.\end{aligned}\tag{14}$$

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- *A clear a priori error bound!*