HOMEWORK 10 **DUE: Wed., June 2** **NAME:**

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will lose points if you do not circle your answers**.

Question	Points	Score
1	2	
2	2	
3	6	
Total	10	

Problem 1: (2 points) Verify the following

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}.$$

Using the definition of the Laplace transform, we find

$$\mathcal{L}\{\cos t\} = \int_0^\infty e^{-st}\cos t dt.$$

By using integration by parts twice we obtain

$$\mathcal{L}\{\cos t\} = s - s^2\mathcal{L}\{\cos t\}.$$

Rearranging, we find

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2+1}.$$

Problem 2: (2 points) Find the inverse Laplace Transform of

$$F(s) = \frac{3s}{s^2-s-6}.$$

Consider

$$F(s) = \frac{3s}{s^2-s-6} = \frac{A}{s-3} + \frac{B}{s+2},$$

by partial fractions and we find $A = 9/5$ and $B = 6/5$. Hence

$$f(t) = \frac{9}{5}e^{3t} + \frac{6}{5}e^{-2t}.$$

Problem 3: (6 points) Use the Laplace Transform to solve the following initial value problem(s)

(a) (2 points) $y'' - y' - 6y = \cos t,\ y(0) = 1,\ y'(0) = -1.$

Taking the Laplace Transform of both sides and using the initial conditions we obtain

$$(s^2 - s - 6)Y(s) - s + 2 = \frac{s}{s^2+1}.$$

Hence

$$Y(s) = \frac{s-2}{s^2-s-6} + \frac{s}{(s^2+1)(s^2-s-6)} = \frac{44}{50}\left(\frac{1}{s+2}\right) + \frac{13}{50}\left(\frac{2}{s-3}\right) - \frac{9}{50}\left(\frac{s}{s^2+1}\right) + \frac{1}{50}\left(\frac{1}{s^2+1}\right),$$

by partial fractions. Hence

$$y(t) = \frac{44}{50}e^{-2t} + \frac{13}{50}e^{3t} + \frac{1}{50}\sin t - \frac{9}{50}\cos t.$$

(b) (2 points) $y'' + 4y = \sin(t) - u_{2\pi}(t)\sin(t-2\pi),\ y(0) = 0,\ y'(0) = 0.$

Taking the Laplace Transform of both sides and using the initial conditions we obtain

$$(s^2+4)Y(s) = \frac{1}{s^2+1}\left(1 - e^{-2\pi s}\right).$$

Hence

$$Y(s) = \frac{1}{(s^2+1)(s^2+4)}\left(1 - e^{-2\pi s}\right) = H(s)\left(1 - e^{-2\pi s}\right),$$

which implies that $y(t) = h(t) - u_{2\pi}h(t-2\pi)$, so we need only find the inverse Laplace transform of $H(s)$.

$$H(s) = \frac{1}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2} + \frac{Cs+D}{s^2+1},$$

and by partial fraction we find $A = C = 0$, $B = -1/3$, and $D = 1/3$. Hence

$$H(s) = -\frac{1}{6}\left(\frac{2}{s^2+4}\right) + \frac{2}{6}\left(\frac{1}{s^2+1}\right),$$

and

$$h(t) = -\frac{1}{6}\left(\sin(2t) - 2\sin(t)\right).$$

Notice that $h(t-2\pi) = h(t)$, which implies that

$$y(t) = -\frac{1}{6}\left(1 - u_{2\pi}(t)\right)\left(\sin(2t) - 2\sin(t)\right).$$

(c) (2 points) $y'' + 4y = \delta(t-\pi) - \delta(t-2\pi),\ y(0) = 0,\ y'(0) = 0.$

Taking the Laplace Transform of both sides and using the initial conditions we obtain

$$(s^2+4)Y(s) = e^{-\pi s} - e^{-2\pi s} \Longrightarrow Y(s) = \frac{1}{2}\left(\frac{2}{s^2+4}\right)\left(e^{-\pi s} - e^{-2\pi s}\right) = H(s)\left(e^{-\pi s} - e^{-2\pi s}\right).$$

Hence $h(t) = \frac{1}{2}\sin(2t)$ and

$$y(t)\frac{1}{2}u_{\pi}(t)\sin\left(2(t-\pi)\right) - \frac{1}{2}u_{2\pi}\sin\left(2(t-2\pi)\right).$$