HOMEWORK 10 **DUE: Wed., Mar. 10** **NAME:**

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will lose points if you do not circle your answers**.

Question	Points	Score
1	2	
2	1	
3	2	
4	2	
5	3	
Total	10	

Problem 1: (2 points) Verify Green's theorem for $\mathbf{F} = -x^2y\mathbf{i} + xy^2\mathbf{j}$, where D is the disk $x^2 + y^2 \leq 4$.

First we note that D is simple and $P = -x^2y$, $Q = xy^2$ are both C^1 on D. Hence, Green's theorem will apply. Green's Theorem states that

$$\int_{\partial\mathbf{D}} Pdx + Qdy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy.$$

A clockwise parametrization of ∂D is $\mathbf{c}(t) = (2\cos t, 2\sin t)$ with $t \in [0, 2\pi]$. Then

$$\int_{\partial\mathbf{D}} Pdx + Qdy = 32\int_0^{2\pi} \cos^2 t \sin^2 t dt = 32\int_0^{2\pi} \sin^2 t\left(1 - \sin^2 t\right) dt = 32\left[\int_0^{2\pi} \sin^2 t dt - \int_0^{2\pi} \left(\sin^2 t\right)^2 dt\right].$$

Using the formula $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$ the integral becomes

$$32\left[\frac{1}{2}\int_0^{2\pi} (1 - \cos 2t)\, dt - \frac{1}{4}\int_0^{2\pi} \left(1 - 2\cos 2t + \cos^2 2t\right) dt\right] = 32\left[\pi - \frac{\pi}{2} + \frac{1}{8}\int_0^{2\pi} (1 - \cos 4t)\, dt\right].$$

Using the formula $\cos^2 2t = \frac{1}{2}(1 + \cos 4t)$, the integral becomes

$$32\left[\frac{\pi}{2} + \frac{1}{8}\int_0^{2\pi} (1 + \cos 4t)\, dt\right] = 32\left[\frac{\pi}{2} - \frac{\pi}{4}\right] = 8\pi.$$

Now looking at the right hand side of greens theorem we find

$$\int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int\int_D \left(x^2+y^2\right) dA = \int_0^2 \int_0^{2\pi} r^3 dr d\theta = 2\pi \int_0^2 r^3 dr = 8\pi.$$

Q.E.D.

Problem 2: (1 point) Let $P(x,y) = -y/\left(x^2+y^2\right)$ and $Q = x/\left(x^2+y^2\right)$. Assuming D is the unit disk, investigate why Green's theorem fails for this P and Q.

Green's theorem is not applicable because it requires P and Q to be differentiable on the domain $D = \left\{(x,y) \in \mathcal{R}^2 | x^2+y^2 = 1\right\}$. However, we can see that these functions are not even bounded, so they certainly cannot be differentiable at the origin $(0,0) \in D$. To show they are not bounded, consider the limit as $(x,y) \to (0,0)$ along the y-axis for P (which is $-\infty$).

Problem 3: (2 points) Use Green's theorem to find the area between the ellipse $x^2/9 + y^2/4 = 1$ and the circle $x^2+y^2 = 25$.

We parametrize the outer boundary, the circle, in a positive, or counter-clockwise, motion, so that the normal is outward to the circle and the boundary to inner boundary, the ellipse, in a negative, or clockwise direction. That is

$$\partial D = \left\{ \begin{array}{lcll} \mathbf{c}_1(t) & = & (5\cos t, 5\sin t) & \text{for } \ t \in [0,2\pi], \\ \mathbf{c}_2(t) & = & (3\cos t, -2\sin t) & \text{for } \ t \in [0,2\pi]. \end{array} \right.$$

Then the area of D is given by

$$A = \frac{1}{2}\left(\int_{\mathbf{c}_1} xdy - ydx + \int_{\mathbf{c}_2} xdy - ydx\right) = \frac{1}{2}\left[25\int_0^{2\pi} dt - 6\int_0^{2\pi} dt\right) = 19\pi.$$

Problem 4: (2 points) Verify Stokes's theorem for the surface defined by $x^2+y^2+5z = 1$ where $z \geq 0$, oriented by an upward normal for the vector field

$$\mathbf{F} = \left(xz, yz, x^2+y^2\right).$$

We first note that $\partial S = \left\{(x,y) \in \mathcal{R}^2 | x^2+y^2 = 1\right\}$ and $\mathbf{F}$ is continuous and differentiable on the S, ∂S, and the domain $D = \left\{(x,y) \in \mathcal{R}^2 | x^2+y^2 \leq 1\right\}$. Since we may write

$$z = f(x,y) = \frac{1}{5}\left(1-x^2-y^2\right),$$

the surface is a graph and the upward facing normal is given by

$$\mathbf{N} = \left(\frac{2}{5}x, \frac{2}{5}y, 1\right).$$

Stokes theorem states that

$$\int\int_D \nabla \times \mathbf{F} \cdot d\mathbf{S} = \oint_{\partial S} \mathbf{F} \cdot d\mathbf{s}.$$

Considering the left hand side, calculating $\nabla \times \mathbf{F} = (y, -x, 0)$, we see that

$$\int\int_D \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int\int_D (\nabla \times \mathbf{F}) \cdot \mathbf{N} dA = \int\int_D 0 dA = 0.$$

Looking at the right hand side, we can parametrize the boundary of S with $\mathbf{c}(t) = (\cos t, \sin t, 0)$ for $t \in [0, 2\pi]$. Hence $\mathbf{c}'(t) = (-\sin t, \cos t, 0)$. So $\mathbf{F}(\mathbf{c}(t)) = (0, 0, 1)$. Hence

$$\oint_{\partial S} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{c}} \mathbf{F}\left(\mathbf{c}(t)\right) \cdot \mathbf{c}'(t) dt = 0.$$

Q.E.D.

Problem 5: (3 points) Let S be the surface defined by $y = 10 - x^2 - z^2$ with $y \geq 1$, oriented with a rightward pointing normal. Let

$$\mathbf{F} = \left(2xyz + 5z, e^x \cos\left(yz\right), x^2 y\right).$$

Determine

$$\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

(Hint: You will need to use an indirect approach.)

First we note that the surface, S, is a graph such that $y = f(z, x) = 10 - x^2 - z^2$ with $y \geq 1$ such that the $\partial S = \left\{(z, x) \in \mathcal{R}^2 | x^2 + z^2 = 9\right\}$. This means that we have a rightward pointing normal of $\mathbf{N} = (2z, 2x, 1)$. Now, we may define a new surface S' such that $y = 1$ and $x^2 + z^2 = 9$ (i.e. the disc at $y = 1$ of radius 3) such that $\partial S' = \partial S$. Then the normal to S' is simply $\mathbf{n} = \mathbf{j}$. Hence by Stokes's theorem

$$\int\int_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int\int_{S'} \nabla \times \mathbf{F} \cdot d\mathbf{S} = \int\int_{S'} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

Calculating $\nabla \times \mathbf{F} = \left(x^2 + ye^x \sin xz, 5, e^x \cos yz - 2xz\right)$, the integral becomes

$$\int\int_{S'} 5 dx dy = 45\pi.$$