HOMEWORK 2 **DUE: Friday, Jan. 15** **NAME:**

DIRECTIONS:

- Attach this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order.
- When required, **do not forget the units**!
- Circle your final answers. **You will loose points if you do not circle your answers**.

Question	Points	Score
1	1	
2	4	
3	2	
4	2	
5	1	
Total	10	

Problem 1: (1 point) Consider the function $f(x, y) = 4 - x^2 - y^2$. Use level curves to construct the graph of the function . You should be able to do this without a graphing calculator. Show the level curves as well as the full graph of f! (Hint: This becomes trivial when write the function f in cylindrical coordinates).

To graph the level curves, consider the surface given by $c = 4 - x^2 - y^2$.

1. Let $c = 0$. Hence $x^2 + y^2 = 4$.
2. Let $c = 4$. Then $x^2 + y^2 = 0 \Longrightarrow (x, y) = (0, 0)$.
3. Let $c = 3$. Hence $x^2 + y^2 = 1$.
4. Let $c = -5$. Then $x^2 + y^2 = 9$.
5. etc.

Problem 2: (4 points) In this problem you will establish rigorously that

$$lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} = 0.$$

(a) (1 points) Show that $|x| \leq ||(x, y)||$ and $|y| \leq ||(x, y)||$.

Consider $|x| = (x^2)^{1/2}$. Squaring both sides we get

$$|x|^2 = x^2 \leq x^2 + y^2.$$

Now taking the square root of both sides

$$|x| \leq (x^2 + y^2)^{1/2} = ||(x, y)||.$$

Similarly $|y| \leq ||(x, y)||$. Q.E.D.

(b) (1 points) Show that $|x^3 + y^3| \leq 2(x^2 + y^2)^{\frac{3}{2}}$. (Hint: Begin with the triangle inequality, and then use part (a).)

The triangle inequality tells us that $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$. So for real numbers, this literally means that

$$\left[(x+y)^2\right]^{1/2} \leq (x^2)^{1/2} + (y^2)^{1/2} \Longrightarrow |x + y| \leq |x| + |y|.$$

Setting $x \to x^3$ and $y \to y^3$ we obtain

$$|x^3 + y^3| \leq |x|^3 + |y|^3 \leq ||(x, y)||^3 + ||(x, y)||^3 = 2||(x, y)||^3 = 2(x^2 + y^2)^{3/2}.$$

Q.E.D.

(c) (1 points) Show that if $||(x, y)|| \leq \delta$, then $\left|\frac{x^3+y^3}{x^2+y^2}\right| \leq 2\delta$.

Part (b) implies that for $(x, y) \neq (0, 0)$

$$\frac{|x^3 + y^3|}{|x^2 + y^2|} \leq \frac{2|x^2 + y^2|^{3/2}}{|x^2 + y^2|} = 2\left(x^2 + y^2\right)^{1/2} \leq 2\delta.$$

Q.E.D.

(d) (1 points) Now prove that $lim_{(x,y)\to(0,0)} \frac{x^3+y^3}{x^2+y^2} = 0$.

Let $\epsilon > 0$ be given. We want to show that $\left|\frac{x^3+y^3}{x^2+y^2}\right| < \epsilon$. Let us assume that

$$||(x, y) - (0, 0)|| < \delta.$$

Choose $\delta = \epsilon/2$. Then

$$||(x, y)|| < \epsilon/2.$$

From part (c), we know that

$$\left|\frac{x^3 + y^3}{x^2 + y^2}\right| < 2 \cdot \frac{\epsilon}{2} = \epsilon.$$

Q.E.D.

Problem 3: (2 points) Find the equation of the plane tangent to the graph $z = e^{x+y}\cos(xy)$ at the point $(0, 1, e)$.

The equation of a plane is given by

$$\frac{\partial f}{\partial x}(x_0, y_0)(x - x_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)(y - y_0) - (z - z_0).$$

Let $f(x,y) = z = e^{x+y}\cos(xy)$. Then we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= e^{x+y}\left(cos(xy) - ysin(xy)\right), \\ \frac{\partial f}{\partial y} &= e^{x+y}\left(cos(xy) - xsin(xy)\right).\end{aligned}$$

So at the point $(0,1,e)$ we have

$$\begin{aligned}\frac{\partial f}{\partial x} &= e, \\ \frac{\partial f}{\partial y} &= e.\end{aligned}$$

Hence, the equation of the plane tangent to the graph of f at $(0,1,e)$ is

$$ex + ey - z = 0.$$

Problem 4: (2 points) Suppose the "Amazing Steve" is fired from a cannon at the angle θ with initial velocity $\mathbf{v}_0 = v_0 cos\theta\mathbf{i} + v_0 sin\theta\mathbf{j}$. Ignore air resistance so that the only force acting on Steve after time $t = 0$ is gravity. Describe his trajectory with a parametric equation $\mathbf{x}(t)$. What is the geometry of his trajectory (i.e. what kind of curve is it)?

Let us align our axis such that the mouth of the canon lines up with the origin. Since the only force acting on Steve after $t = 0$ is gravity, we know his acceleration for $t > 0$ is given by

$$\mathbf{a}(t) = -g\mathbf{j},$$

where $g = 9.8\text{m/s}^2$ is the acceleration due to gravity. Let us denote the path Steve makes though the air as $\mathbf{x}(t)$. We know that $\mathbf{x}'(t) = \mathbf{v}(t)$ where $\mathbf{v}(t)$ is the velocity and also that $\mathbf{x}''(t) = \mathbf{a}(t)$. So the antiderivative of $\mathbf{a}(t)$ is the velocity. Hence

$$\mathbf{v}(t) = \left(-gt + c_1\right)\mathbf{j} + c_2\mathbf{i},$$

where c_1 and c_2 are constants. But we know Steve's initial velocity so

$$\mathbf{v}(0) = c_1\mathbf{j} + c_2\mathbf{i} = \mathbf{v}_0 = v_0 cos\theta\mathbf{i} + v_0 sin\theta\mathbf{j}.$$

Hence $c_1 = v_0 sin\theta$ and $c_2 = v_0 cos\theta$. Similarly, the path $\mathbf{x}(t)$ is the antiderivative of the velocity

$$\mathbf{x}(t) = \left(-\frac{gt^2}{2} + v_0 t sin\theta + d_1\right)\mathbf{j} + \left(v_0 t cos\theta + d_2\right)\mathbf{i},$$

where d_1 and d_2 are constants. But since we aligned the mouth of the cannon with the origin, we know $\mathbf{x}(0) = \mathbf{0}$, so $d_1 = d_2 = 0$. Hence the path traced by the "Amazing Steve" is the parabola given by

$$\mathbf{x}(t) = \left(-\frac{gt^2}{2} + v_0 t sin\theta\right)\mathbf{j} + v_0 t cos\theta\mathbf{i}.$$

Problem 5 (1 point) List your project group member names (including your own), email addresses, and phone numbers. Remember groups must consist of 3-4 people.

This must be a list of 3-4 people.