HOMEWORK 3 **DUE: Fri., Apr. 16** **NAME:**

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will lose points if you do not circle your answers**.

Question	Points	Score
1	1	
2	3	
3	2	
4	1	
5	1	
6	1	
7	1	
Total	10	

Problem 1: (1 point) Solve the difference equation $y_{n+1} = (-1)^{n+1}y_n$ in terms of an initial value y_0. Describe the behavior of the solution as $n \to \infty$.

$$\begin{aligned} y_0 &= y_0 \\ y_1 &= -y_0 \\ y_2 &= -y_0 \\ y_3 &= y_0 \\ y_4 &= y_0 \\ y_5 &= -y_0 \\ . &= . \\ . &= . \\ . &= . \end{aligned} \tag{1}$$

So the solution is given by

$$y_n = \begin{cases} -y_0, & n = 4k-2, 4k-3 \\ y_0, & n = 4k, 4k-1 \end{cases}$$

for $k \in \mathcal{N}$. This will oscillate for all time.

Problem 2: (3 points) Consider the initial value problem

$$y' = 2xy, \quad y(0) = 1.$$

(a) (1.5 points) Use Picard's method to find a solution to the differential equation $y' = 2xy$. Integrating the differential equation and using the fundamental theorem of calculus yields

$$y(x) = y_0 + \int_{x_0}^{x} 2ty(t)dt,$$

where $(x_0, y_0) = (1, 0)$, hence

$$y(x) = 1 + \int_0^x 2ty(t)dt.$$

Let $\phi_0 = y_0 = 1$ and

$$\phi_n = 1 + \int_0^x 2t\phi_{n-1}(t)dt,$$

for $n \in \{1, 2, 3, ...\}$. Then each ϕ_n satisfies the initial condition, but none satisfy the differential equation. Iterating, we find

$$\begin{aligned}
\phi_0 &= 1 \\
\phi_1 &= 1 + x^2 \\
\phi_2 &= 1 + x^2 + \frac{x^4}{2} \\
\phi_3 &= 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{6} \\
. &= . \\
. &= . \\
. &= . \\
\phi_n &= \sum_{k=0}^{n} \frac{x^{2k}}{k!} \qquad (2)
\end{aligned}$$

so each ϕ_n exists. Taking the limit

$$\lim_{n\to\infty} \phi_n = e^{x^2}.$$

So $y(x) = e^{x^2}$. Note, this satisfies both the initial condition and the differential equation.

(b) (1.5 points) Prove that this solution is unique.

Suppose the solution $y(x) = \phi(x)$ is not unique. That is, suppose there exists both $\phi(x)$ and $\psi(x)$ such that

$$\begin{aligned}
\phi(x) &= 1 + \int_0^x 2t\phi(t)dt, \\
\psi(x) &= 1 + \int_x^x 2t\psi(t)dt.
\end{aligned} \qquad (3)$$

Then the absolute value of the difference is

$$|\phi(x) - \psi(x)| = \left| \int_0^x 2t\left(\phi(t) - \psi(t)\right) dt \right| \leq \int_0^x 2t\left|\phi(t) - \psi(t)\right| dt,$$

for $t \geq 0$. Let us look at a rectangle containing (x_0, y_0). Without loss of generality, suppose we restrict $x \in [0, A/2]$ for some $A \in \mathcal{R}$. Then

$$|\phi(x) - \psi(x)| \leq A \int_0^x |\phi(t) - \psi(t)|\, dt.$$

Define $U(x) = \int_0^x |\phi(t) - \psi(t)|\, dt$. Then it is clear that $U(x) \geq 0$ and, in fact, $U(0) = 0$. Therefore, from the fundamental theorem of calculus, $U'(x) = |\phi(x) - \psi(x)|$. This implies that

$$U'(x) \leq AU(x) \Longrightarrow U'(x) - AU(x) \leq 0.$$

Multiplying both sides by e^{-Ax} and integrating yields

$$\int_0^x \frac{d}{dx}\left(e^{-Ax}U(x)\right) dx \leq \int_0^x 0dx.$$

Hence $e^{-Ax}U(x \leq 0 \Longrightarrow U(x) \leq 0$. Hence we must have $U(x) = 0 \Longrightarrow \psi(x) = \phi(x)$ and the solution must be unique. Q.E.D.

Problem 3: (2 points) Solve the given equation

$$\left(3x^2y + 2xy + y^3\right) dx + \left(x^2 + y^2\right) dy = 0.$$

(Hint: Consider using an integration factor.)

Define

$$\begin{aligned} M(x,y) &= 3x^2y + 2xy + y^3, \\ N(x,y) &= x^2 + y^2. \end{aligned}$$

It is clear that $M_y \neq N_x$ so the equation is not exact. However, if we multiply both sides by $\mu(x,y)$ then if

$$(\mu M)_y = (\mu N)_x\,,$$

the new equation will be exact. Suppose μ is a function of x only. Then μ must satisfy

$$\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu = 3\mu.$$

Using separation of variables, we find $\mu = e^{3x}$. So the equation

$$\mu M dx + \mu N dy = 0,$$

is exact, hence

$$\psi(x,y) = \int \mu M dx = x^2ye^{3x}_{\frac{y^3}{3}}e^{3x} + g(y).$$

But we know $\psi_y = N$ hence $g(y) = C$. Therefore the solution is given by

$$\psi(x,y) = e^{3x}\left(x^2y + \frac{y^3}{3}\right) = C.$$

Problem 4: (1 point) Solve the initial value problem

$$y'' - y' - 2y = 0, \quad y(0) = \alpha, \quad y'(0) = 2.$$

Find the value of α such that the solution approaches 0 as $t \to \infty$.

This is a linear, homogeneous, equation with constant coefficients so the characteristic equation is

$$m^2 - m - 2 = 0 \Longrightarrow m_1 = 2,\ m_2 = -1.$$

So the two parameter family of solutions is given by

$$y(x) = c_1 e^{2x} + c_2 e^{-x}.$$

The initial conditions are $y(0) = \alpha$ and $y'(0) = 2$ which yield

$$\begin{aligned} c_1 + c_2 &= \alpha, \\ 2c_1 - c_2 &= 2. \end{aligned} \tag{4}$$

Solving this system of equations yields $c_1 = \frac{\alpha+2}{3}$ and $c_2 = 2\left(\frac{\alpha-1}{3}\right)$. So the solution to the initial value problem is given by

$$y(x) = \left(\frac{\alpha+2}{3}\right) e^{2x} + 2\left(\frac{\alpha-1}{3}\right) e^{-x}.$$

In order for this solution to remain finite as $x \to \infty$ we must have $\frac{\alpha+2}{3} = 0 \Longrightarrow \alpha = -2$.

Problem 5: (1 point) If the Wronskian, W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

$$W = \begin{vmatrix} e^{2t} & g(t) \\ 2e^{2t} & g'(t) \end{vmatrix} = 2e^{4t}.$$

This yields the differential equation

$$g' - 2g = 2e^{2t},$$

which has the solution

$$g(t) = 3te^{2t} + Ce^{2t}.$$

Problem 6: (1 point) Find the solution of the initial value problem

$$y'' + 2y' + 2y = 0, \quad y(\pi/4) = 2, \quad y'(\pi/4) = -2.$$

This is a linear, homogeneous equation with constant coefficients. Hence, the characteristic equation is given by

$$m^2 + 2m + 2 = 0 \Longrightarrow m = -1 \pm i.$$

Hence, the general solution is given by

$$y(x) = e^{-x}\left(c_1 \cos x + c_2 \sin x\right).$$

The initial conditions imply that

$$\begin{aligned} c_1 + c_2 &= 2\sqrt{2}e^{\pi/4}, \\ c_1 &= \sqrt{2}e^{\pi/4}. \end{aligned}$$

So the solution to the initial value problem is

$$y(x) = \sqrt{2}e^{\pi/4}e^{-x}\left(\cos x + \sin x\right).$$

Problem 7: (1 point) Use the method of reduction of order to find a second solution of

$$(x-1)\,y'' - xy' + y = 0, \quad x > 0,$$

where $y_1(x) = e^x$.

The solution is given by $y_2(x) = v(x)y_1(x) = v(x)e^x$. Plugging this into the differential equation above implies that v must solve

$$v'' + \left(1 - \frac{1}{x-1}\right) v' = 0.$$

Letting $u = v'$, this becomes the first order equation

$$u' + \left(1 - \frac{1}{x-1}\right) u = 0.$$

Separation of variables yields

$$u(x) = (x-1)e^{-x}.$$

Since $v' = u$, integrating both sides shows $v = -xe^{-x}$. So the second solution is given by

$$y_2 = vy_1 = -x,$$

but since the constant out front is arbitrary, $y_2 = x$.