HOMEWORK 3 **DUE: Friday, Jan. 22** **NAME:**

DIRECTIONS:

- Attach this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will loose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will loose points if you do not circle your answers**.

Question	Points	Score
1	2	
2	2	
3	4	
4	2	
Total	10	

Problem 1: (2 point) Suppose $\mathbf{x}$ and $\mathbf{y}$ are differentiable paths in three-space (i.e. $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))$ and $\mathbf{y}(t) = (y_1(t), y_2(t), y_3(t))$). Show that

$$\frac{d}{dt}(\mathbf{x} \times \mathbf{y}) = \frac{d\mathbf{x}}{dt} \times \mathbf{y} + \mathbf{x} \times \frac{d\mathbf{y}}{dt}.$$

First consider the left hand side (L.H.S.).

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \mathbf{i}\,(x_2y_3 - x_3y_2) - \mathbf{j}\,(x_1y_3 - x_3y_1) + \mathbf{k}\,(x_1y_2 - x_2y_1)\,. \tag{1}$$

Now take the derivative and we have

$$\begin{aligned} \tfrac{d}{dt}(\mathbf{x} \times \mathbf{y}) \quad = \quad & +\mathbf{i}\,(x_2'y_3 + x_2y_3' - x_3'y_2 - x_3y_2') \\ & -\mathbf{j}\,(x_1'y_3 + x_1y_3' - x_3'y_1 - x_3y_1') \\ & +\mathbf{k}\,(x_1'y_2 + x_1y_2' - x_2'y_1 - x_2y_1')\,. \end{aligned} \tag{2}$$

Now consider the right hand side (R.H.S). We know that $\frac{d\mathbf{x}}{dt} = x_1'\mathbf{i} + x_2'\mathbf{j} + x_3'\mathbf{k}$ and $\frac{d\mathbf{y}}{dt} = y_1'\mathbf{i} + y_2'\mathbf{j} + y_3'\mathbf{k}$, hence

$$\frac{d\mathbf{x}}{dt} \times \mathbf{y} = \mathbf{i}\,(x_2'y_3 - x_3'y_2) - \mathbf{j}\,(x_1'y_3 - x_3'y_1) + \mathbf{k}\,(x_1'y_2 - x_2'y_1)\,,$$

and

$$\mathbf{x} \times \frac{d\mathbf{y}}{dt} = \mathbf{i}\,(x_2y_3' - x_3y_2') - \mathbf{j}\,(x_1y_3' - x_3y_1') + \mathbf{k}\,(x_1y_2' - x_2y_1')\,,$$

Sum the two and we have exactly the L.H.S. Q.E.D.

Problem 2: (2 points) In electrostatics, the force $\mathbf{P}$ of attraction between two particles of opposite charge is given by $\mathbf{P} = k\left(\mathbf{r}/||\mathbf{r}||^3\right)$, is called *Coulomb's Law*, where k is a constant and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that $\mathbf{P}$ is the gradient of $f = -k/||\mathbf{r}||$.

Consider $f = \frac{-k}{||\mathbf{r}||} = \frac{-k}{\sqrt{x^2+y^2+z^2}}$. Then

$$\bigtriangledown f = k\left(\frac{x}{[x^2+y^2+z^2]^{3/2}}, \frac{y}{[x^2+y^2+z^2]^{3/2}}, \frac{z}{[x^2+y^2+z^2]^{3/2}}\right) = k\frac{\mathbf{r}}{||\mathbf{r}||^3}.$$

Q.E.D.

Problem 3: (4 points) Captain Ralph (grandson of the Amazing Steve) is in trouble near the sunny side of Mercury. The temperature of the ship's hull when he is at location (x, y, z) is given by $T(x, y, z) = e^{-x^2-2y^2-3z^2}$, where x, y, and z are measured in meters. He is currently at $(1, 1, 1)$.

(a) (2 points) In what direction should he proceed in order to decrease the temperature most rapidly?

$$\bigtriangledown T = \left(-2xe^{-x^2-2y^2-3z^2}, -4ye^{-x^2-2y^2-3z^2}, -4ze^{-x^2-2y^2-3z^2}\right)$$

So at the point $(1, 1, 1)$ the gradient is

$$\bigtriangledown T|_{(1,1,1)} = \left(-2e^{-6}, -4e^{-6}, -6e^{-6}\right) \text{deg/m},$$

and is the direction Ralph should proceed in to decrease the temperature most rapidly.

(b) (2 points) If the ship travels at e^8 meters per second, how fast will the temperature decrease if he proceeds in that direction?

We are looking for the speed so we want

$$|| \bigtriangledown T_{(1,1,1)}|| \times e^8 \text{deg/s} = 2\sqrt{14}e^2.$$

Problem 4 (2 points) The three-dimensional *heat equation* is the partial differential equation

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \frac{\partial T}{\partial t},$$

where k is constant. It models the temperature $T(x, y, z, t)$ at the point (x, y, z) and time t of a body in space.

(a) (1.5 points) Show that $T(x, y, z, t) = e^{-kt}(cosx + cosy + cosz)$ satisfies the three dimensional heat equation.

Taking the second partial derivatives we find

$$\begin{aligned}
\frac{\partial^2 T}{\partial x^2} &= -e^{-kt}cosx, \\
\frac{\partial^2 T}{\partial y^2} &= -e^{-kt}cosy, \\
\frac{\partial^2 T}{\partial z^2} &= -e^{-kt}cosz, \\
\frac{\partial T}{\partial t} &= -ke^{-kt}(cosx + cosy + cosz).
\end{aligned}$$

Plugging these into the three-dimensional heat equation shows that T does, in fact, satisfy this equation.

(b) (0.5 point) Describe what happens to the temperature of the body after a long period of time.

Over long periods of time, we can think of what happens as $t \to \infty$. In this case, we know that $cosx + cosy + cosz \leq 3$ at all times and $e^{-kt} \to 0$ as $t \to \infty$. Hence, the temperature $T \to 0$ as $t \to \infty$. That is, the temperature of the body goes to or approaches zero quickly and is very small after a long period of time.