HOMEWORK 4 **DUE: Fri., Apr. 23** **NAME:**

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will lose points if you do not circle your answers**.

Question	Points	Score
1	3	
2	2	
3	4	
4	1	
Total	10	

Problem 1: (3 point) Solve the initial value problem given by

$$y'' + y' - 2y = 2t,$$

where $y(0) = 0$ and $y'(0) = 1$.

First we solve the homogeneous problem with characteristic equation given by

$$m^2 + m - 2 = (m-1)(m+2) = 0,$$

Which has roots $m = 1, -2$. Hence the complementary equation is

$$y_c = c_1 e^t + c_2 e^{-2t}.$$

We will use the method of undetermined coefficients to solve for the particular solution y_p. Hence

$$y_p = At + B.$$

We note that nothing of this sort appears in the complementary solution, y_c, implying that this is a good guess for the particular solution. Plugging this into the differential equation yields $A = -1$ and $B = -1/2$. Hence the particular solution is given by $y_p = -t - 1/2$. The two parameter family of solutions is then

$$y = c_1 e^t + c_2 e^{-2t} - t - 1/2.$$

The initial conditions yield $c_1 = 1$ and $c_2 = 1/2$. Hence

$$y = e^t - \frac{1}{2}e^{-2t} - t - \frac{1}{2}.$$

Problem 2: (2 points) Let D be the operator given by $D[y] = \frac{\partial y}{\partial x}$. Show that the operator $(xD-1)(D+4)$ is not the same as the operator $(D+4)(xD-1)$.

First consider $(xD-1)(D+4)$ applied to y

$$(xD-1)(D+4)[y] = (xD-1)[y'+4y] = xy'' - y' + 4xy' - 4y = xy'' + (4x-1)y' - 4y.$$

Now consider$(D+4)(xD-1)$ applied to y

$$(D+4)(xD-1)[y] = (D+4)[xy'-y] = y' + xy'' + 4xy' - 4y = xy'' + (4x+1)y' - 4y.$$

Q.E.D.

Problem 3: (4 points) Solve the differential equation

$$y'' - y = \cosh t,$$

using

(a) (1.5 point) variation of parameters.

The homogeneous equation is

$$y'' - y = 0,$$

which has the characteristic equation $m^2 - 1 = 0$ which implies the complimentary solution is

$$y_c = c_1 e^t + c_2 e^{-t}.$$

Using variation of parameters, this means that

$$y_p = u_1(t)e^t + u_2(t)e^{-t}.$$

The Wronskian is $W = -2$, hence

$$\begin{aligned} u_1' &= \frac{1}{4}\left(1 + e^{-2t}\right) \\ u_2' &= -\frac{1}{4}\left(1 + e^{2t}\right) \end{aligned}$$

recalling that $\cosh x = \frac{1}{2}\left(e^t + e^{-t}\right)$. Integrating yields

$$\begin{aligned} u_1 &= \frac{1}{4}\left(t - \frac{1}{2}e^{-2t}\right) \\ u_2 &= -\frac{1}{4}\left(t + \frac{1}{2}e^{2t}\right) \end{aligned}$$

Hence the particular solution is given by

$$y_p = \frac{1}{4}te^t - \frac{1}{4}te^{-t} \underbrace{-\frac{1}{8}e^t - \frac{1}{8}e^{-t}}_{\text{absorb into } y_c}$$

so

$$y = c_1 e^t + c_2 e^{-t} + \frac{1}{4}te^t - \frac{1}{4}te^{-t}.$$

(b) (1.5 point) undetermined coefficients.

The homogeneous equation is

$$y'' - y = 0,$$

which has the characteristic equation $m^2 - 1 = 0$ which implies the complimentary solution is

$$y_c = c_1 e^t + c_2 e^{-t}.$$

Using undetermined coefficients we would guess

$$y_p = Ae^t + Be^{-t},$$

but both of these are already accounted for in the complimentary solution so we guess again that

$$y_p = Ate^t + Bte^{-t}.$$

Plugging this into the differential equation and solving gives $A = 1/4$ and $B = -1/4$ hence

$$y_p = \frac{1}{4}te^t - \frac{1}{4}te^{-t},$$

hence

$$y = c_1 e^t + c_2 e^{-t} + \frac{1}{4}te^t - \frac{1}{4}te^{-t}.$$

(c) (1 point) Are the solutions the same in (a) and (b)? Could you have predicted this? Justify your answer.

Yes, the solutions are the same as we would have expected because if we write the equation as

$$y'' + p(t)y' + q(t)y = g(t),$$

then we see that p, q, and g are all continuous everywhere so as long as we find a two parameter family of solutions, it would have to contain all solutions, implying that the method we use to solve it should not matter.

Problem 4: (1 point) Find the solution of the differential equation

$$y'' - 2y' - 3y = g(t),$$

where $g(t)$ is an arbitrary continuous function.

The homogeneous equation is

$$y'' - 2y' - 3y = 0,$$

with characteristic equation $m^2 - 2m - 3 = 0$. Hence the complimentary solution is

$$y_c = c_1 e^{3t} + c_2 e^{-t}.$$

Since we know that $g(t)$ is continuous, if we are able to find a two parameter family of solutions, it will have to provide a formula for all possible solutions. The two independent solutions $y_1 = e^{3t}$ and $y_2 = e^{-t}$ yield the Wronskian, $W(y_1, y_2) = -4e^{2t}$. Then using variation of parameters

$$\begin{aligned} u_1' &= \frac{1}{4}e^{-3t}g(t), \\ u_2' &= -\frac{1}{4}e^{t}g(t). \end{aligned}$$

Hence

$$
\begin{aligned}
u_1(t) &= \int_{t_0}^{t} e^{-3s} g(s)ds, \\
u_2(t) &= \int_{t_0}^{t} e^{s} g(s)ds.
\end{aligned}
$$

So the full solution is given by

$$y = c_1 e^{3t} + c_2 e^{-t} + u_1 y_1 + u_2 y_2 = c_1 e^{3t} + c_2 e^{-t} + \frac{1}{4} e^{3t} \int_{t_0}^{t} e^{-3s} g(s)ds - \frac{1}{4} e^{-t} \int_{t_0}^{t} e^{s} g(s)ds.$$