HOMEWORK 4 **DUE: Friday, Jan. 29** **NAME:**

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will loose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will loose points if you do not circle your answers**.

Question	Points	Score
1	1	
2	2	
3	3	
4	3	
5	1	
Total	10	

Problem 1: (1 point) Calculate the second-order Taylor polynomial for $f(x, y) = \cos x \cos y$ at the point $(0, \pi/2)$.

The Taylor polynomial is given by

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \sum_{i=1}^{n} h_i \frac{\partial f}{\partial x_i}(\mathbf{x}_0) + \frac{1}{2} \sum_{i,j=1}^{\infty} h_i h_j \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{x}_0) + R_2(\mathbf{x}_0 . \mathbf{h}).$$

So calculating the partials of f

$$\begin{array}{rclcrcl}
\frac{\partial f}{\partial x} & = & -\sin x \cos y & \Longrightarrow & \frac{\partial f}{\partial x}(\mathbf{x}_0) & = & 0, \\
\frac{\partial f}{\partial y} & = & -\cos x \sin y & \Longrightarrow & \frac{\partial f}{\partial y}(\mathbf{x}_0) & = & -1, \\
\frac{\partial^2 f}{\partial x^2} & = & -\cos x \cos y & \Longrightarrow & \frac{\partial^2 f}{\partial x^2}(\mathbf{x}_0) & = & 0, \\
\frac{\partial^2 f}{\partial y^2} & = & -\cos x \cos y & \Longrightarrow & \frac{\partial^2 f}{\partial y^2}(\mathbf{x}_0) & = & 0, \\
\frac{\partial^2 f}{\partial x \partial y} & = & \sin x \sin y & \Longrightarrow & \frac{\partial^2 f}{\partial x \partial y}(\mathbf{x}_0) & = & 0.
\end{array}$$

Hence

$$f(\mathbf{x}_0 + \mathbf{h}) = -h_2.$$

Problem 2: (2 points) A metal plate has the shape of the region $x^2 + y^2 \leq 1$. The plate is heated so that the temperature at any point (x, y) on it is indicated by

$$T(x, y) = 2x^2 + y^2 - y + 3.$$

Fine the hottest and coldest points on the plate, and the temperature at each of these points (Hint: Parametrize the boundary of the plate in order to find any critical points there.)

Let us define $U = \{\mathbf{x} \in \mathbb{R}^2 \mid ||\mathbf{x}|| \leq 1\}$ as the disc centered at the origin of radius 1. Then the boundary $\partial U = \{\mathbf{x} \in \mathbb{R}^2 \mid ||\mathbf{x}|| = 1\}$. Let us first consider the critical points of T (not restricted to U). These are given when

$$\mathbf{D}f(\mathbf{x}) = \mathbf{0},$$

that is, when

$$\left(\frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}\right) = \mathbf{0} \Longrightarrow \left\{ \begin{array}{rclcrcl} 4x & = & 0 & \Longrightarrow & x & = & 0, \\ 2y - 1 & = & 0 & \Longrightarrow & y & = & \frac{1}{2}. \end{array} \right.$$

Notice, the point $\mathbf{x}_0 = \left(0, \frac{1}{2}\right) \in U - \partial U$ (i.e. in U, but not in the ∂U). Now let us consider the boundary, which can be parametrized by

$$\mathbf{c}(t) = (\sin t, \cos t),$$

with $0 \leq t \leq 2\pi$. To locate the critical values of T on ∂U, it suffices to locate the maxima and minima of

$$g(t) = T(\mathbf{c}(t)) = 2\sin^2 t + \cos^2 t - \cos t + 3.$$

Setting $\frac{dg}{dt} = 0$, we find

$$\sin t\,(2\cos t - 1) = 0 \Longrightarrow \left\{ \begin{array}{rclcrcl} \sin t & = & 0 & \Longrightarrow & t & = & 0,\ \pi \\ or & & & & & & \\ \cos t & = & \frac{1}{2} & \Longrightarrow & t & = & \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}. \end{array} \right.$$

So examining the values of T for critical values internal to U, we have

$$T\left(0, \frac{1}{2}\right) = 2.74,$$

and for critical values on ∂U, we have

$$\begin{array}{rcl} t = 0 & \Longrightarrow & T(0,1) = 3, \\ t = \pi & \Longrightarrow & T(0,-1) = 5, \\ t = \dfrac{\pi}{3} & \Longrightarrow & T\left(\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right) = 4.25, \\ t = 2\pi - \dfrac{\pi}{3} & \Longrightarrow & T\left(\dfrac{-\sqrt{3}}{2}, \dfrac{1}{2}\right) = 4.25. \end{array}$$

Hence the maximum is achieved at the point $(0, -1)$ on the boundary, and the minimum is achieved at the point $\left(0, \frac{1}{2}\right)$ in the interior of U.

Problem 3: (3 points) Suppose the cone $z^2 = x^2 + y^2$ is sliced by the plane $z = x + y + 2$ so that a conic section C is created. Use Lagrange multipliers to find the points on C that are nearest to and farthest from the origin. (Hint: Think about the shape of C. What does it look like?).

The problem is to find the minimum and maximum distances from $(0, 0, 0)$ of points (x, y, z) on C. For algebraic simplicity, we may look at the square of the distance, rather than the actual distance. Thus, we desire to find the extrema of

$$f(x, y, z) = x^2 + y^2 + z^2,$$

subject to the constraints

$$\begin{aligned} g_1(x,y,z) &= x^2+y^2-z^2=0, \\ g_2(x,y,z) &= x+y-z=-2. \end{aligned}$$

The constrained critical points (x_0,y_0,z_0) must statisfy

$$\bigtriangledown f(x_0,y_0,z_0)=\lambda_1 \bigtriangledown g_1(x_0,y_0,z_0)+\lambda_2 \bigtriangledown g_2(x_0,y_0,z_0),$$

as well as the two constraint equations. Thus, we must solve

$$\begin{aligned} 2x &= 2\lambda_1 x+\lambda_2, \\ 2y &= 2\lambda_1 y+\lambda_2, \\ 2z &= -2\lambda_1 z-\lambda_2, \\ x^2+y^2-z^2 &= 0, \\ x+y-z &= -2. \end{aligned}$$

Eliminating λ_2 from the first two equations yields

$$\lambda_2=2x-2\lambda_1 x=2y-2\lambda_1 y,$$

which implies

$$2(x-y)(1-\lambda_1)=0.$$

Therefore either $x=y$ or $\lambda_1=1$. The condition $\lambda_1=1$ implies immediately that $\lambda_2=0$, and the third equation in the system becomes $2z=-2z \Longrightarrow z=0$, then $x=y=0$ from the fourth equation. However, $(0,0,0)$ is not a point on the plane $z=x+y+2$. Thus, the condition $\lambda_1=1$ leads to no critical points. On the other hand, if $x=y$, then th constraint equations (the last two) become

$$\begin{aligned} 2x^2-z^2 &= 0, \\ 2x-z &= -2. \end{aligned}$$

Substituting $z=2x+2$ yields

$$2x^2+8x+4=0,$$

whose solutions are $x=-2\pm\sqrt{2}$. Therefore, there are two constrained points:

$$\begin{aligned} \mathbf{a}_1 &= \left(-2+\sqrt{2},-2+\sqrt{2},-2+2\sqrt{2}\right), \\ \mathbf{a}_2 &= \left(-2-\sqrt{2},-2-\sqrt{2},-2-2\sqrt{2}\right). \end{aligned}$$

We can check that $f(\mathbf{a}_1)=24-16\sqrt{2}$ and $f(\mathbf{a}_2)=24+16\sqrt{2}$. At first glance, it would seem that $\mathbf{a}_1$ must be the point on C lying nearest the origin and $\mathbf{a}_2$ must be the point that lies farthest. However, we don't know a priori if there is a farthest point from the origin. To understand what kind of curve C is, note that $\mathbf{a}_1$ has a positive z-coordinate and $\mathbf{a}_2$ has a negative z-coordinate. Therefore, the plane $z=x+y+2$ intersects both nappes of the cone, meaning C must be a hyperpola. This means that $\mathbf{a}_1$ is, indeed the point nearest the origin, but $\mathbf{a}_2$ is not the farthest point.

Problem 4: (3 points) Find the critical points of $f(x,y)=x^2+y$ subject to $x^2+2y^2=1$ and use the Hessian criterion to determine the nature of the critical point(s).

Let h be the auxiliary function such that $h(x,y,\lambda)=f(x,y)-\lambda g(x,y)$ where $g(x,y)=x^2+2y^2-1$. Then

$$h(x,y,\lambda)=x^2+y-\lambda(x^2+2y^2-1).$$

The critical points occur when the following are satisfied

$$\begin{aligned} 2x(1-\lambda) &= 0, \\ 1-4\lambda y &= 0, \\ x^2+2y^2-1 &= 0, \end{aligned}$$

The first equation implies that either $x=0$ or $\lambda=1$. If $x=0$ then we find the critical points are given by $\mathbf{c}_{1,2}=(x,y,\lambda)=\left(0,\pm\frac{\sqrt{2}}{2},\pm\frac{\sqrt{2}}{4}\right)$ and if $\lambda=1$, the critical points are given by $\mathbf{c}_{3,4}=(x,y,\lambda)=\left(\pm\sqrt{\frac{7}{8}},\frac{1}{4},1\right)$. The bordered Hessian for this problems is given by

$$|\bar{\mathbf{H}}| = \begin{vmatrix} 0 & -2x & -4y \\ -2x & 2(1-\lambda) & 0 \\ -4y & 0 & -4\lambda \end{vmatrix} = 16x(x\lambda-2y^2)-32y^2(1-\lambda).$$

At the points $\mathbf{c}_{1,2}$, $|\bar{\mathbf{H}}|<0$ implying these are relative minima, while at the points $\mathbf{c}_{3,4}$, $|\bar{\mathbf{H}}|>0$, implying these are relative maxima.

Problem 5: (1 point) Consider the equations that relate polar and Cartesian coordinates:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

These equations define x and y as functions or r and θ. Use the Inverse Function theorem to determine the set of points $\{\mathbf{x}\}$ near which we can invert these equations. What can you say about the inverse function theorem at the origin?

We first need to compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

Thus, we see that, away from the origin (r=0), we can solve (locally) for r and θ uniquely in terms of x and y. At the origin, however, the inverse function theorem does not apply. Geometrically, this makes perfect sense, since at the origin, the polar angle θ can have any value.