HOMEWORK 5 **DUE: Fri., Apr. 30** **NAME:**

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will lose points if you do not circle your answers**.

Question	Points	Score
1	2	
2	2	
3	2	
4	2	
5	2	
Total	10	

Problem 1: (2 point) Consider the differential equation

$$y^{(4)} + 2y''' + y'' = 0.$$

Verify that the functions $y_1(t) = 1$, $y_2(t) = t$, $y_3(t) = e^{-t}$, and $y_4(t) = te^{-t}$ are solutions to the equation and determine their Wronskian.

$y_1(t) = 1$ is a solution since $y_1' = 0$,
$y_2(t) = t$ is a solution since $y_2'' = 0$,
$y_3(t) = e^{-t}$ is a solution since $y_3' = -e^{-t},\ y_3'' = e^{-t},\ y_3''' = -e^{-t},\ y_3^{(4)} = e^{-t}$,
$y_4(t) = te^{-t}$ is a solution since $y_4' = (1-t)e^{-t},\ y_4'' = (-2+t)e^{-t},\ y_4''' = (3-t)e^{-t},\ y_4^{(4)} = (-4+t)e^{-t}$.

The Wronskian is given by

$$W = \begin{vmatrix} 1 & t & e^{-t} & te^{-t} \\ 0 & 1 & -e^{-t} & (1-t)e^{-t} \\ 0 & 0 & e^{-t} & (-2+t)e^{-t} \\ 0 & 0 & -e^{-t} & (3-t)e^{-t} \end{vmatrix} = e^{-2t} \neq 0,$$

for any t.

Problem 2: (2 points) Let the linear differential operator L be defined by

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y,$$

where $a_0, a_1, ..., a_n \in \mathcal{R}$.

(a) (1 point) Find $L[t^n]$.

$$L[t^n] = \sum_{k=0}^{n} a_k \frac{n!}{k!} t^k.$$

(b) (1 point) Find $L[e^{mt}]$.

$$L[e^{mt}] = e^{mt} \sum_{k=0}^{n} a_k m^{n-k}.$$

Problem 3: (2 point) Determine four solutions of the equation $y^{(4)} - 5y'' + 4y = 0$. Do you think the four solutions form a fundamental set of solutions? Explain your answer.

The characteristic equation is given by

$$m^4 - 5m + 4 = 0.$$

This may be factored as

$$\left(m^2 - 4\right)\left(m^2 - 1\right) = 0.$$

Hence the solution is given by

$$y(t) = c_1 e^t + c_2 e^{-t} + c_3 e^{2t} + c_4 e^{-2t}.$$

Since $W(e^t, e^{-t}, e^{2t}, e^{-2t}) \neq 0$ these form a fundamental set of solutions.

Problem 4: (2 points) Find the general solution to the differential equation

$$y^{(5)} + 5y^{(4)} - 2y^{(3)} - 10y^{(2)} + y^{(1)} + 5y = 0.$$

The characteristic equation is given by

$$m^5 + 5m^4 - 2m^3 - 10m^2 + m + 5 = 0.$$

The possible rational roots of this equation are $\pm\frac{1}{1}$, $\pm\frac{5}{1}$. Plugging these possibilities into the equation, we find that $+1, -1, -5$ are, in fact, roots. Using synthetic division we factor the equation to obtain

$$(m-1)^2(m+1)^2(m+5) = 0.$$

Hence, the solution is given by

$$y(t) = c_1 e^t + c_2 t e^t + c_3 e^{-t} + c_4 t e^{-t} + c_5 e^{-5t}.$$

Problem 5: (2 point) Solve the initial value problem

$$y''' - 8y = 0,$$

subject to $y(0) = 0, \quad y'(0) = -1, \quad y''(0) = 0.$

The characteristic equation is given by
$$m^3 - 8 = 0,$$
which factors as
$$(m-2)(m^2+2m+4) = 0.$$
Hence the general solution is
$$y(t) = c_1 e^{2t} + e^{-t}\left(c_2 \cos\sqrt{3}t + c_3 \sin\sqrt{3}t\right).$$
The initial conditions yield the following system of equations
$$\begin{array}{rcl} c_1 + c_2 & = & 0, \\ 2c_1 - c_2 + \sqrt{3}c_3 & = & -1, \\ 4c_1 - 2c_2 - 2\sqrt{3}c_3 & = & 0. \end{array}$$
Solving this system yields $c_1 = -1/6$, $c_2 = 1/6$, and $c_3 = -\sqrt{3}/6$. Hence the solution to the initial value problem is given by
$$y = -\frac{1}{6}e^{2t} + e^{-t}\left(\frac{1}{6}\cos\sqrt{3}t - \frac{\sqrt{3}}{6}\sin\sqrt{3}t\right).$$