HOMEWORK 5 **DUE: Friday, Feb. 5** **NAME:**

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will loose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will loose points if you do not circle your answers**.

Question	Points	Score
1	1	
2	2	
3	2	
4	2	
5	3	
Total	10	

Problem 1: (1 point) Recall the "Amazing Steve" from your second Homework Assignment. What is the force which which he hits the ground after his 'flight'?

Steve's acceleration at all times was $\mathbf{a} = -g\mathbf{j}$, hence, from Newton's second law

$$\mathbf{F} = m\mathbf{a} = -gm\mathbf{j},$$

where m is Steve's mass.

Problem 2: (2 points) Suppose Steve is fired form the cannon with an angle of inclination of $\theta = 45$ degrees and that he hits the ground 500 meters from the cannon. What, then, was Steve's initial speed?

Recall that the path Steve travels is given by

$$\mathbf{x}(t) = v_0 t \cos\theta \mathbf{i} + \left(-\frac{gt^2}{2} + v_0 t \sin\theta\right)\mathbf{j}.$$

The we know at time $t = t_f$ (the final time, when he hits the ground) we must have $\mathbf{x}(t_f) = (500, 0)$ meters. Hence we have

$$\begin{aligned} v_0 t_f \cos\theta &= 500, \\ t_f\left(-\frac{gt_f}{2} + v_0 \sin\theta\right) &= 0. \end{aligned}$$

The second implies that either $t_f = 0$ (which cannot be the final time since it is the initial time) or $-\frac{gt_f}{2} + v_0 \sin\theta = 0$. Combining this with the first (and substituting $\theta = \pi/4$) yields

$$v_0 = \sqrt{500g} \text{ m/s}.$$

So his initial velocity is

$$\mathbf{v} = \sqrt{250g}\ \mathbf{i} + \sqrt{250g}\ \mathbf{j}$$

in meters per second.

Problem 3: (2 points) Let $\mathbf{c}(t)$ be a path, $\mathbf{v}(t)$ its velocity, and $\mathbf{a}(t)$ the acceleration. Suppose $\mathbf{F}$ is a C^1 mapping of $\mathbb{R}^3$ to $\mathbb{R}^3$, $m > 0$, and $\mathbf{F}(\mathbf{c}(t)) = m\mathbf{a}(t)$ (Newton's second law). Prove that

$$\frac{d}{dt}\left[m\mathbf{c}(t) \times \mathbf{v}(t)\right] = \mathbf{c}(t) \times \mathbf{F}(\mathbf{c}(t)).$$

What can you conclude if $\mathbf{F}(\mathbf{c}(t))$ is parallel to $\mathbf{c}(t)$?

Consider the left hand side

$$\frac{d}{dt}\left[m\mathbf{c}(t) \times \mathbf{v}(t)\right] = m\left[\mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{c}(t) \times \mathbf{a}(t)\right].$$

But $\mathbf{v}(t) \times \mathbf{v}(t) = 0$, hence

$$\frac{d}{dt}\left[m\mathbf{c}(t) \times \mathbf{v}(t)\right] = m\mathbf{c}(t) \times \mathbf{a}(t) = m\mathbf{c}(t) \times \mathbf{F}(\mathbf{c}(t)).$$

Q.E.D.

Additionally, if $\mathbf{F}(\mathbf{c}(t))$ is parallel to $\mathbf{c}(t)$, then $\mathbf{c}(t) \times \mathbf{F}(\mathbf{c}(t)) = 0$ which means that the tangent vector crossed with the path is always constant, implying that the path stay's in a plane (i.e. a subset of $\mathbb{R}^2$).

Problem 4: (2 points) Define the unit tangent vector T of the path x as the normalization of the velocity vector; that is,

$$\mathbf{T} = \frac{\mathbf{x}'(t)}{||\mathbf{x}'(t)||}.$$

Prove that $\frac{d\mathbf{T}}{dt}$ is perpendicular to $\mathbf{T}$ for all time, t.

Consider the norm of T squared

$$||\mathbf{T}||^2 = \mathbf{T}\dot{\mathbf{T}} = \frac{\mathbf{x}'(t) \cdot \mathbf{x}'(t)}{||\mathbf{x}'(t)||} = 1.$$

So we know that $||\mathbf{T}||^2 = 1$. Hence

$$\frac{d}{dt}(\mathbf{T} \cdot \mathbf{T}) = 0 \Longrightarrow \mathbf{T}' \cdot \mathbf{T} = 0.$$

Hence, $\mathbf{T}$ and $\mathbf{T}'$ are always perpendicular. Q.E.D.

Problem 5: (3 points) Consider the helix $\mathbf{x}(t) = (a\cos t, a\sin t, bt)$, $0 \le t \le 2\pi$.

(a) (1 point) What is the total length, $L(\mathbf{x})$, of the path?

The total length of the path is given by

$$L(\mathbf{x}) = \int_0^{2\pi} ||\mathbf{x}(t)||dt = \int_0^{2\pi} \sqrt{a^2+b^2}dt = 2\pi\sqrt{a^2+b^2}.$$

(b) (2 points) Define a new parameter $s = t\sqrt{a^2+b^2}$. Suppose we define the curvature κ of a path $\mathbf{x}$ as the angular rate of change in the direction of $\mathbf{T}$ per unit change in distance along the path. That is, define

$$\kappa = \frac{||d\mathbf{T}/dt||}{ds/dt}.$$

What is the curvature of the helix?

We first must calculate the unit tangent vector

$$\mathbf{T} = \frac{\mathbf{x}'}{||\mathbf{x}'||} = \frac{1}{\sqrt{a^2+b^2}}\left(-a\sin t, a\cos t, b\right).$$

Hence

$$\mathbf{T}' = \frac{-1}{\sqrt{a^2+b^2}}\left(a\cos t, a\sin t, 0\right).$$

Therefore the curvature is given by

$$\kappa = \frac{||d\mathbf{T}/dt||}{ds/dt} = \frac{\frac{a}{\sqrt{a^2+b^2}}}{\sqrt{a^2+b^2}} = \frac{a}{a^2+b^2}.$$