HOMEWORK 6 **DUE: Wed., Feb. 10** **NAME:**

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will lose points if you do not circle your answers**.

Question	Points	Score
1	2	
2	4	
3	2	
4	2	
Total	10	

Problem 1: (2 points) Prove that for a vector field, $\mathbf{F} = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$, the divergence of the curl is zero. That is, prove

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0.$$

Consider first

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \mathbf{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \mathbf{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \mathbf{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right).$$

Hence

$$\nabla \cdot (\nabla \times \mathbf{F}) = \frac{\partial^2 F_3}{\partial x \partial y} - \frac{\partial^2 F_2}{\partial x \partial z} - \frac{\partial^2 F_3}{\partial y \partial x} + \frac{\partial^2 F_1}{\partial y \partial z} + \frac{\partial^2 F_2}{\partial z \partial x} - \frac{\partial^2 F_1}{\partial z \partial y} = 0.$$

Q.E.D.

Problem 2: (4 points) Determine which of the following vector fields is *not* a gradient vector field.

(a) (2 point) $\mathbf{F} = \left(x^2 + y^2\right)\mathbf{i} - 2xy\mathbf{j}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y^2 & -2xy & 0 \end{vmatrix} = -4y\mathbf{k} \neq 0,$$

so $\mathbf{F}$ is not a gradient vector field.

(b) (2 point) $\mathbf{F} = 3x^2y\mathbf{i} + (x^3 + y^3)\mathbf{j}$.

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2y & x^3 + y^3 & 0 \end{vmatrix} = 0,$$

so $\mathbf{F}$ is a gradient vector field.

Problem 3: (2 points) Evaluate the following integral over the rectangle R given by $[0, 2] \times [-1, 0]$

$$\int\int_R \left[|y| \cos\left(\frac{\pi x}{4}\right)\right] dydx.$$

Evaluating the integral yields

$$\int_0^2 \int_- 1^0 \left[-y \cos\left(\frac{\pi x}{4}\right)\right] dydx = 0.$$

Problem 4: (2 points) Although Fubini's theorem holds for most functions we'll see in practice, you still need to be careful. For example, you can show that

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dydx = \frac{\pi}{4},$$

yet

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} dxdy = -\frac{\pi}{4}.$$

Why does this not contradict Fubini's theorem?

Firstly, recall that Fubini's theorem requires the function to be continuous or at the very least bounded. Clearly the function

$$f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2},$$

does not exist at the point $(0, 0)$ in the rectangle R give by $[0, 1] \times [0, 1]$. Moreover, consider the nature of f as it approaches $(0, 0)$ along the x-axis (i.e. $y = 0$),

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{(x^2 + y^2)^2} = \lim_{x\to 0} \frac{1}{x^2} = \infty,$$

and along the y-axis (i.e. $x = 0$)

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{(x^2 + y^2)^2} = \lim_{y\to 0} -\frac{1}{y^2} = -\infty.$$

So clearly f is unbounded.