HOMEWORK 7 **DUE: Fri., May 14** **NAME:** Key

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order.**
- When required, **do not forget the units!**
- Circle your final answers. **You will lose points if you do not circle your answers.**

Question	Points	Score
1	1.5	
2	2	
3	2.5	
4	2	
5	2	
Total	10	

Problem 1: (1.5 points) Consider the following differential equation

$$x^2\,(x+1)^2\,y'' + \left(x^2-1\right)y' + 2y = 0.$$

(a) (0.5 points) Identify the ordinary points.

The ordinary points are all real numbers except $x_0 = 1, -1$.

(b) (1 point) Identify and classify the singular points.

The singular points are $x_0 = 0, -1$. $x_0 = 0$ is an irregular singular point since

$$\lim_{x\to 0} \frac{x^2-1}{x^2(x+1)^2}\cdot x \to \infty.$$

$x_0 = -1$ is a regular singular point since

$$\lim_{x\to -1} \frac{x^2-1}{x^2(x+1)^2}\cdot (x+1) = -2 < \infty,$$

and

$$\lim_{x\to -1} \frac{2}{x^2(x+1)^2}\cdot (x+1)^2 = 2 < \infty.$$

Problem 2: (2 points) Find the indicial roots of the following differential equation. What can you say about the certainty of getting two linearly independent solutions if you were to apply Frobenius's Method?

$$x(x-1)y'' + 3y' - 2y = 0.$$

We may easily verify that $x_0 = 0$ is a regular singular point so we suppose a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}.$$

Plugging this into the differential equation yields

$$\sum_{n=0}^{\infty} c_n(n+r)(n+r)x^{n+r} - \sum_{n=0}^{\infty} c_n(n+r)(n+r-1)x^{n+r-1} + 3\sum_{n=0}^{\infty} c_n(n+r)x^{n+r-1} - 2\sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

Letting $k = n-1$ in the second and third summation and $k = n$ in the first and fourth, we find

$$\sum_{k=0}^{\infty} c_k(k+r)(k+r)x^{k+r} - \sum_{k=-1}^{\infty} c_{k+1}(k+1+r)(k+r)x^{k+r} + 3\sum_{k=-1}^{\infty} c_{k+1}(k+1+r)x^{k+r} - 2\sum_{k=0}^{\infty} c_k x^{k+r} = 0.$$

Grouping terms and simplifying, we find

$$x^r\left\{\frac{c_0}{x}\left(-r(r-1)+3r\right) + \sum_{k=0}^{\infty}\left[c_{k+1}(k+r+1)(-k-r+3) + c_k\left((k+r)(k+r-1)-2\right)\right]x^k\right\} = 0.$$

This yields the indicial equation

$$-r(r-1) + 3r = -r(r-4) = 0.$$

Hence, the indicial roots are $r = 0$, and $r = 4$, which differ by an integer. Therefore, would would not necessarily expect Frobenius's method to yeild two distinct solutions.

Problem 3: (2.5 points) Use the method of Frobenius to find two linearly independent series solutions about the regular singular point $x_0 = 0$ for the following differential equation.

$$2xy'' - y' + 2y = 0.$$

Again, $x_0 = 0$ can easily be shown to be a regular singular point so we assume a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}.$$

Plugging this into the differential equation and simplifying yields

$$x^r\left\{\frac{c_0}{x}r(2r-3) + \sum_{k=0}^{\infty}\left[c_{k+1}(k+r+1)(2k+2r-1) + 2c_k\right]x^k\right\} = 0.$$

This yields the indicial equation

$$r(2r-3) = 0 \Longrightarrow r_1 = 0,\ r_2 = \frac{3}{2}.$$

These indicial roots do not differ by an integer, so we can expect Frobenius's method to yield two linearly independent solutions. First suppose $r = r_1 = 0$. Then the recursion relation is give by

$$c_{k+1} = \frac{-2c_k}{(k+1)(2k-1)} \text{ for } k = 0, 1, 2, 3, \ldots.$$

Hence

$$\begin{aligned} c_1 &= -2c_0, \\ c_2 &= \frac{(-2)^2 c_0}{2!}, \\ c_3 &= \frac{(-2)^3 c_0}{3 \cdot 3!}, \\ c_4 &= \frac{(-2)^4 c_0}{3 \cdot 5 \cdot 4!}, \\ c_5 &= \frac{(-2)^5 c_0}{3 \cdot 5 \cdot 7 \cdot 5!}, \\ . &= . \\ . &= . \\ . &= . \\ c_n &= \frac{(-2)^n c_0}{n!(3 \cdot 5 \cdots (2n-3))}, \text{ for } n = 2, 3, 4, ... \end{aligned}$$

Hence the first solution is given by

$$y_1 = c_0 \left[1 - 2x + \sum_{n=2}^{\infty} \frac{(-2)^n c_0}{n!(3 \cdot 5 \cdots (2n-3))} x^n \right].$$

Now suppose $r = r_2 = 3/2$. Then

$$c_{k+1} = \frac{c_k}{(k+1)(k+2)}.$$

Hence

$$\begin{aligned} c_1 &= \frac{c_0}{2}, \\ c_2 &= \frac{c_0}{2 \cdot 3!}, \\ c_3 &= \frac{c_0}{3!4!}, \\ . &= . \\ . &= . \\ . &= . \\ c_n &= \frac{c_0}{n!(n+1)!}, \text{ for } n = 1, 2, 3, ... \end{aligned}$$

So the second solution is given by

$$y_2 = c_0 \left[1 + \sum_{n=1}^{\infty} \frac{c_0}{n!(n+1)!} x^{n+\frac{3}{2}} \right].$$

Problem 4: (2 points) Suppose $z = 1 + i$ and $w = 3 - \sqrt{2}i$.

(a) (0.5 points) Compute $z \cdot w$.

$$z \cdot w = \left(3 + \sqrt{2}\right) + i\left(3 - \sqrt{2}\right).$$

(b) (0.5 points) Compute z/w.

$$z/w = \frac{(3 - \sqrt{2}) + i\,(3 + \sqrt{2})}{11}.$$

(c) (0.5 points) Write z in terms of its modulus and Argument.

$$z = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right).$$

(d) (0.5 points) Calculate the roots of $u^4 = z$. The roots are given by

$$u_k = |z|^{1/4}\left(\cos\theta_k + i\sin\theta_k\right),$$

where $\theta_k = \frac{\pi}{16} + k\frac{\pi}{2}$. Hence

$$\begin{aligned}\theta_0 &= \frac{\pi}{16},\\ \theta_1 &= \frac{9\pi}{16},\\ \theta_2 &= -\frac{15\pi}{16},\\ \theta_3 &= -\frac{7\pi}{16}.\end{aligned}$$

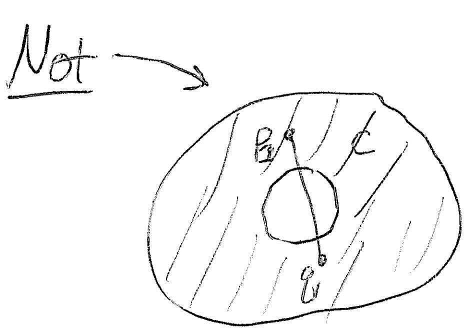
Hence, the roots are given by

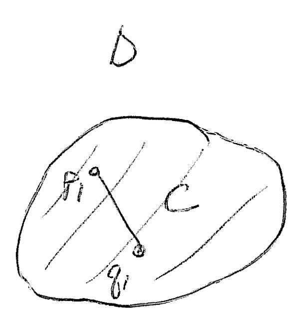
$$\begin{aligned}u_0 &= 2^{1/8}\left(\cos\left(\frac{\pi}{16}\right) + i\sin\left(\frac{\pi}{16}\right)\right),\\ u_1 &= 2^{1/8}\left(\cos\left(\frac{9\pi}{16}\right) + i\sin\left(\frac{9\pi}{16}\right)\right),\\ u_2 &= 2^{1/8}\left(\cos\left(\frac{-15\pi}{16}\right) + i\sin\left(\frac{-15\pi}{16}\right)\right),\\ u_3 &= 2^{1/8}\left(\cos\left(\frac{-7\pi}{16}\right) + i\sin\left(\frac{-7pi}{16}\right)\right).\end{aligned}$$

Problem 5: (2 points) Let C be a bounded, closed, convex set and let D be the complement of C. Show that D is a domain. You may use a pictorial proof.

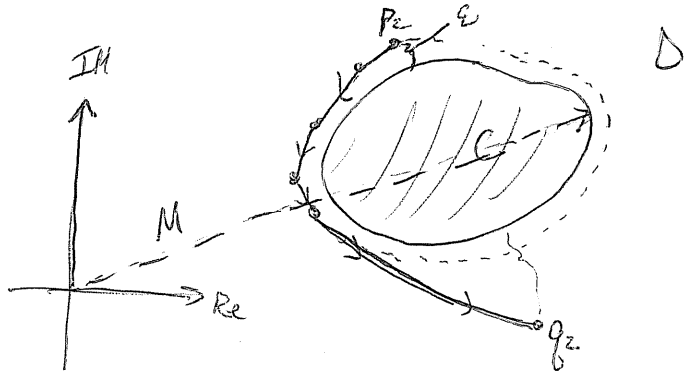
Proof: Because C is closed, I know that $D = C^c$ must be open (proof in text). Therefore we need only show that D is connected in order to show that D is a domain. We will use a pictorial proof to show that we can construct a polygonal path between any two points in D.

Because C is convex, we know that for all points p_1 and q_1 in C, the line segment p_1q_1 is completely contained in C (see picture below). This means that C has no "holes" in it (i.e. it is a solid region rather than some sort of "donut").





Therefore, all points of D lie outside the boundary of C (note that the boundary of C, ∂C, is the same as the boundary of D, ∂D). Let p_2, q_2 be two points in D. Let $\epsilon = \min\{$ distance from p_2 to ∂C, distance q_2 to $\partial C\}$. Because C is bounded, that means that there exists an $M \in \mathcal{R}^+$ such that $|z| < M$ for all $z \in C$. See picture below.



This means that I can draw a "path" around C which is always at least ϵ distance from ∂C from p_2 to q_2 consisting of straight line segments. Anytime a segment is about to come within ϵ of ∂C, we change directions to follow the boundary until the direct line towards q_2 is contained entirely in D.