**HOMEWORK 7** **DUE: Fri., Feb. 19** **NAME:**

**DIRECTIONS:**

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will lose points if you do not circle your answers**.

| Question | Points | Score |
|---|---|---|
| 1 | 2 | |
| 2 | 3 | |
| 3 | 2 | |
| 4 | 3 | |
| Total | 10 | |

**Problem 1:** (2 points) Compute the volume of the solid bounded by the surface $z = \sin y$, the planes $x = 1$, $x = 0$, $y = 0$, and $y = \pi/2$, and the xy plane.

$$\int_0^1 \int_0^{\pi/2} \sin y dy dx = \int_0^1 -\cos y|_0^{\pi/2} dx = \int_0^1 dx = 1.$$

**Problem 2:** (3 points) Let D be the region bounded by the positive x and y axes and the line $12x+4y = 12$. Compute

$$\int\int_D \left(x^2 + y^2\right) dA.$$

The solution is given by

$$\int\int_D \left(x^2 + y^2\right) dA = \int_0^1 \int_0^{-3x+3} \left(x^2 + y^2\right) dy dx = \int_0^1 \left(x^2 y + \frac{y^3}{3}\right) |_0^{-3x+3} dx.$$

With a little bit of algebraic manipulation, this becomes

$$\int_0^1 \left[3x^2 - 3x^3 + 9\left(1 - x\right)^3\right] dx = \frac{5}{2}.$$

**Problem 3:** (2 points)

**(a)** Prove the Mean Value Theorem for Double Integrals. That is, suppose $f : D \to \mathbb{R}$ is continuous and $D$ is an elementary region. The for some point $(x_0, y_0)$ in $D$ we have

$$\int\int_D f(x,y)dA = f(x_0,y_0)A(D),$$

where $A(D)$ is the area of $D$.

**Proof:** Because $f$ is continuous on $D$, it attains its maximum, which we will call $M$, for some $(x_1, y_1) \in D$ and its minimum, which we will call $m$, for some $(x_2, y_2) \in D$. Hence we know for all $(x, y) \in D$,

$$m \leq f(x,y) \leq M.$$

Now the mean value inequality gives us

$$m \cdot A(D) \leq \int\int_D f(x,y)dA \leq M \cdot A(D),$$

where $A(D)$ is the area of the region $D$. Dividing through this inequality by the area $A(D)$, we see that

$$m \leq \frac{1}{A(D)} \int\int_D f(x,y)dA \leq M.$$

Now because this integral is bounded between the maximum and minimum of $f$, and because we know $f$ is continuous, we know that it must take on every value between $m$ and $M$. Hence, there must exist an $(x_0, y_0) \in D$ such that $f$ takes on just the right value such that

$$\int\int_D f(x,y)dA = f(x_0,y_0)A(D).$$

**(b)** Use the mean value theorem to show that if $D = [-1, 1] \times [-1, 2]$, then

$$1 \leq \int\int_D \frac{1}{x^2+y^2+1} dxdy \leq 6.$$

Consider the function $f(x,y) = \frac{1}{x^2+y^2+1}$. This function is the largest when the denominator is the smallest. (Note: this function is always positive). Hence $x^2 + y^2 + 1$ is smallest when $(x, y) = (0, 0)$. Hence the maximum of $f$ is

$$M = f(0,0) = 1.$$

Similarly, $f$ is smallest when the denominator is largest which clearly occurs at $(1, 2)$ or $(-1, 2)$, in which case

$$m = f(1,2) = f(-1,2) = \frac{1}{6}.$$

Now since the area of the domain $D$, is $A(D) = 6$, the Mean Value Inequality yeilds

$$1 \leq \int\int_D \frac{1}{x^2+y^2+1} dxdy \leq 6.$$

**Problem 4:** (3 points) Evaluate the following integrals.

**(a)** (1 point) $\int_0^4 \int_{y/2}^2 e^{x^2} dxdy$

Changing the order of integration we find

$$\int_0^4 \int_{y/2}^2 e^{x^2} dxdy = \int_0^2 \int_0^{2x} e^{x^2} dydx = \int_0^2 2xe^{x^2} dx.$$

By substituting $u = x^2 \Longrightarrow du = 2xdx$ and $x = 0 \to u = 0$, $x = 2 \to u = 4$, we find

$$\int_0^2 2xe^{x^2} dx = \int_0^4 e^u du = e^4 - 1.$$

**(b)** (1 point) $\int_0^1 \int_0^x \int_0^y (y + xz) dzdydx$.

This problem can be easily integrated without changing the order of integration so we will do so.

$$\int_0^1 \int_0^x \int_0^y (y + xz) dzdydx = \int_0^1 \int_0^x \left( yz + \frac{xz^2}{2} \right) |_0^y dydx = \int_0^1 \int_0^x \left( y^2 + \frac{xy^2}{2} \right) dydx.$$

Integrating again we obtain

$$\int_0^1 \left( \frac{x^3}{3} + \frac{x^4}{6} \right) dx = \frac{1}{12} + \frac{1}{30}.$$

**(c)** (1 point) $\int \int \int_W zdxdydz$, where $W$ is the region bounded by $x + y + z = a$ (with $a > 0$), $x = 0$, $y = 0$, and $z = 0$. This integral can be written as

$$\int_0^a \int_0^{a-x} \int_0^{a-x-y} zdzdydx = \frac{1}{2} \int_0^a \int_9 = 0^{a-x} \left[ (x - a) + y \right] dydx$$

By integrating (and accounting for the correct number of negative signs)

$$-\frac{1}{6} \int_0^a (x - a)^3 \, dx = \frac{a^4}{24}.$$