HOMEWORK 8 **DUE: Fri., May 21** **NAME:**

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will lose points if you do not circle your answers**.

Question	Points	Score
1	1	
2	2	
3	2	
4	3	
5	2	
Total	10	

Problem 1: (1 point) Find the values of $i^{\sqrt{3}}$ (Note, you needn't restrict yourself to the Principle Branch).

$$i^{\sqrt{3}} = e^{\log\left(i^{\sqrt{3}}\right)} = e^{\sqrt{3}} \log i,$$

where

$$\log i = \frac{i\pi}{2}\left(4n+1\right),$$

for $n = 0, \pm 1, \pm 2, ...$ Hence

$$i^{\sqrt{3}} = e^{\frac{\sqrt{3}i\pi}{2}(4n+1)},$$

where $n = 0, \pm 1, \pm 2, ...$

Problem 2: (2 points) Find the solutions of

$$z^{\pi+i} = 10.$$

$$e^{\log z^{\pi+i}} = e^{\log 10} = \ln 10 + 2\pi i n,$$

for $n = 0, \pm 1, \pm 2, ...$ Hence

$$(\pi + i) \log z = \ln 10 + 2\pi i n.$$

Multiplying both sizes by $\pi - i$ then dividing by $\pi^2 + 1$ we find

$$\log z = x + iy,$$

where

$$\begin{aligned} x &= \frac{\pi(\ln 10 + 2n)}{\pi^2 + 1}, \\ y &= \frac{-\ln 10 + 2\pi^2 n}{\pi^2 + 1}. \end{aligned}$$

Hence $z = e^x(\cos y + i \sin y)$.

Problem 3: (2 points) Given the formulae

$$\cosh x = \frac{1}{2}\left(e^x + e^{-x}\right), \text{ and } \sinh x = \frac{1}{2}\left(e^x - e^{-x}\right),$$

for $x \in \mathcal{R}$, prove that

$$\cos z = \cos x \cosh y - i \sin x \sinh y.$$

for $z = x + iy \in \mathcal{C}$.

First we note that

$$\begin{aligned} e^{iz} &= e^{ix-y} = e^{-y}(\cos x + i \sin x), \\ e^{-iz} &= e^{-ix+y} = e^{y}(\cos x - i \sin x). \end{aligned}$$

Hence

$$\cos(x + iy) = \frac{1}{2}\left(e^{iz} + e^{-iz}\right) = \frac{1}{2}\left[\cos x\left(e^y + e^{-y}\right) - i \sin x\left(e^y - e^{-y}\right)\right] = \cos x \cosh y - i \sin x \sinh y.$$

Problem 4: (3 points) Prove that

$$\tan^{-1} z = \frac{i}{2}\mathrm{Log}\left(\frac{1 - iz}{1 + iz}\right),$$

for $z \neq \pm i$.

Let $w = \tan z$, then $z = \tan^{-1} w$.

$$w = \tan z = \frac{\frac{1}{2i}\left(e^{iz} - e^{-iz}\right)}{\frac{1}{2}\left(e^{iz} + e^{-iz}\right)}.$$

Hence

$$iw\left(e^{iz} + e^{-iz}\right) = e^{iz} - e^{-iz}.$$

Multiplying both sizes by e^{iz} and solving for e^{2iz} we find

$$e^{2iz} = \frac{1 + iw}{1 - iw},$$

and solving for z so long as $w \neq i$ yields

$$z = \frac{i}{2}\mathrm{Log}\left(\frac{1 - iw}{1 + iw}\right),$$

(note $-\log(x) = \log x^{-1}$ so long as we make certain $x \neq 0 \iff w \neq -i$). Then replacing w with z we find

$$\tan^{-1} z = \frac{i}{2}\mathrm{Log}\left(\frac{1-iz}{1+iz}\right),$$

for $z \neq \pm i$.

Problem 5: (2 points) Parametrize the circle of radius ϵ centered at $z_0 \in \mathcal{C}$. How does this relate to proving that

$$\int_\gamma \frac{1}{z-z_0} = \begin{cases} 0, & \text{if } z_0 \text{ is outside } \gamma, \\ 2\pi i, & \text{if } z_0 \text{ is inside } \gamma. \end{cases}$$

where γ is a piece-wise smooth, positively oriented, simple closed curve.

We may parametrize the circle $\gamma_1(t) = z_0 + \epsilon e^{it}$ where $t \in [0, 2\pi]$. This is the circle of radius ϵ parametrized in the counterclockwise direction. If we isolate a pole z_0 (see class notes) inside γ_1 which is wholly within γ, we can show

$$\int_\gamma \frac{1}{z-z_0} = \begin{cases} 0, & \text{if } z_0 \text{ is outside } \gamma, \\ 2\pi i, & \text{if } z_0 \text{ is inside } \gamma. \end{cases}$$