HOMEWORK 8 **DUE: Fri., Feb. 26** **NAME:**

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will lose points if you do not circle your answers**.

Question	Points	Score
1	2	
2	2	
3	2	
4	4	
Total	10	

Problem 1: (2 points) Let $\mathbf{c}(t) : [0, 2\pi] \to \mathbb{R}^3$ be the helix given by $\mathbf{c}(t) = (\cos t, \sin t, t)$ and let $f(x, y, z) = xy + z$. Compute the path integral (also called the scalar line integral)

$$\int_{\mathbf{c}} f ds.$$

The path integral is given by

$$\int_{\mathbf{c}} f dx = \int_0^{2\pi} f(\mathbf{c}(t))||\mathbf{c}'(t)||dt.$$

Since $\mathbf{c}(t) = (\cos t, \sin t, t)$, $\mathbf{c}'(t) = (-\sin t, \cos t, 1)$, and $||\mathbf{c}'(t)|| = \sqrt{2}$, the integral becomes

$$\sqrt{2}\int_0^{2\pi} (\sin t \cos t + t)\, dt = \sqrt{2}\int_0^{2\pi} \left[\frac{1}{2}\sin(2t) + t\right] dt = 2\sqrt{2}\pi^2.$$

Problem 2: (2 points) Let $\mathbf{c}(t) : [0, 1] \to \mathbb{R}$ be the path given by $\mathbf{c}(t) = \left(t, t^2, t^3\right)$ and let $\mathbf{F} = y\cos z\mathbf{i} + x\sin z\mathbf{j} + xy\sin z^2\mathbf{k}$. Calculate the line integral

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s}.$$

The line integral is given by

$$\int_{\mathbf{c}} \mathbf{F} \cdot d\mathbf{s} = \int_0^1 \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t)dt.$$

Since $\mathbf{c}'(t) = \left(1, 2t, 3t^2\right)$, and $\mathbf{F}\left(\mathbf{c}(t)\right) = \left(t^2 \cos t^3, t \sin t^3, t^3 \sin t^6\right)$, the integral becomes

$$\int_0^1 t^2 \cos t^3 dt + \int_0^1 2t^2 \sin t^3 dt + \int_0^1 3t^5 \sin t^6 dt.$$

Letting $u = t^3$, $v = t^3$, and $w = t^6$, the integral becomes

$$\frac{1}{3}\int_0^1 \cos u du + \frac{2}{3}\int_0^1 \sin v dv + \frac{1}{2}\int_0^1 \sin w dw = \frac{1}{3}\sin(1) - \frac{7}{6}\left(\cos(1) - 1\right).$$

Problem 3: (2 points) Let $\mathbf{F} = \left(2z^5 - 3xy\right)\mathbf{i} - x^2\mathbf{j} + x^2 z\mathbf{k}$. Calculate the line integral of $\mathbf{F}$ around the perimeter of the square with vertices $(1,1,3)$, $(-1,1,3)$, $(-1,-1,3)$, and $(1,-1,3)$, oriented counterclockwise.

The square with vertices $(1,1,3)$, $(-1,1,3)$, $(-1,-1,3)$, and $(1,-1,3)$, oriented counterclockwise can be parametrized by

$$\mathbf{c}(t) = \begin{cases} \mathbf{c}_1(t) &= & (1-2t, 1, 3) & 0 \le t \le 1, \\ \mathbf{c}_2(t) &= & (-1, -2t+3, 3) & 1 < t \le 2, \\ \mathbf{c}_3(t) &= & (2t-5, -1, 3) & 2 < t \le 3, \\ \mathbf{c}_4(t) &= & (1, 2t-7, 3) & 3 < t \le 4. \end{cases}$$

However, we notice that if we let $\mathbf{x}(t) : [0:2] \to \mathbb{R}^3$ and $\mathbf{y}(t) : [2,4] \to \mathbb{R}^3$ be given by

$$\mathbf{x}(t) = \begin{cases} \mathbf{c}_1(t) &= & (1-2t, 1, 3) & 0 \le t \le 1, \\ \mathbf{c}_2(t) &= & (-1, -2t+3, 3) & 1 < t \le 2. \end{cases}$$

and

$$\mathbf{y}(t) = \begin{cases} \mathbf{c}_3(t) &= & (2t-5, -1, 3) & 2 < t \le 3, \\ \mathbf{c}_4(t) &= & (1, 2t-7, 3) & 3 < t \le 4. \end{cases}$$

Then

$$\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{c}_1} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{c}_3} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{c}_2} \mathbf{F}\cdot d\mathbf{s} = \int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} + \int_{\mathbf{y}} \mathbf{F}\cdot d\mathbf{s}.$$

But calculating each of these shows

$$\int_{\mathbf{x}} \mathbf{F}\cdot d\mathbf{s} = -4\cdot 3^5 + \frac{2}{3},$$

and

$$\int_{\mathbf{y}} \mathbf{F}\cdot d\mathbf{s} = 4\cdot 3^5 - \frac{2}{3},$$

hence

$$\int_{\mathbf{c}} \mathbf{F}\cdot d\mathbf{s} = 0.$$

Problem 4: (4 points) Suppose "Augg of the Hairy Knuckle Clan" (pre-human ancestor of the "Amazing Steve" and "Captain Ralph") is pushing a boulder up a 100 foot-tall spiral path.

(a) (2 points) Suppose Augg's path is described by

$$\mathbf{c}(t) = (5\cos 3t, 5\sin 3t, 10t), \quad 0 \le t \le 10.$$

If he exerts a force with a constant magnitude of 50 lbs tangent to his path, find the work done by Augg in pushing the boulder up to the top of the spiral path.

The force is given by

$$\mathbf{F}\left(\mathbf{c}(t)\right) = 50\frac{\mathbf{c}'(t)}{||\mathbf{c}'(t)||}.$$

So the work, W, done is given by

$$W = \int_0^{10} 50\frac{\mathbf{c}'(t)\cdot\mathbf{c}'(t)}{||\mathbf{c}'(t)||}dt = \int_0^{10} 50||\mathbf{c}'(t)||dt = \int_0^{10} 250\sqrt{13}dt = 2500\sqrt{13}\text{ ft}\cdot\text{lbs.}$$

(b) (2 points) Just as Augg reaches the top of the spiral path, he sneezes and the boulder rolls all the way to the bottom. If the bolder weighs 75 lbs, how much work is done by gravity when the boulder reaches the bottom?

Now the force is 75 lbs in the direction of gravity hence $\mathbf{F} = 75\mathbf{k}$, so the work done is

$$W = \int_0^{10} 75\cdot 10dt = 7500\text{ ft}\cdot\text{lbs.}$$