HOMEWORK 9 **DUE: Fri., May 28** **NAME:** Key

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order.**
- When required, **do not forget the units!**
- Circle your final answers. **You will lose points if you do not circle your answers.**

Question	Points	Score
1	2	
2	1	
3	4	
4	3	
Total	10	

Problem 1: (2 points) Recall that a function $f(z) = u + iv$ is analytic at a point z_0 if it is analytic in some neighborhood about z_0. That is, there exists some $R > 0$ such that f is analytic on $B_R(z_0) = \{z \in \mathbb{C} | |z - z_0| < R\}$. Show that the function $f(z) = (x^2 + y) + i(y^2 - x)$ is not analytic at any point. (Hint: recall that if f is analytic in an open set, then the Cauchy-Riemann equations must hold on that set.)

Here $u(x, y) = x^2 + y$ and $v(x, y) = y^2 - x$. The Cauchy-Riemann equations are

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned}$$

If f is to be analytic at some point z_0, then it must be analytic in the neighborhood $B_R(z_0)$ and the Cauchy-Riemann equations must hold, hence

$$\begin{aligned} 2x &= 2y, \\ 1 &= 1. \end{aligned}$$

But this only holds when $x = y$, and not for all $z \in B_R(z_0)$ for any z_0 or R. Hence f is analytic nowhere.

Problem 2: (1 point) Find the radius of convergence of the power series

$$\sum_{k=0}^{\infty} \frac{z^{3k}}{2^k}.$$

(Hint, you can't use either the ratio test or the root test directly.)

Suppose we let $w = z^3$, then the series becomes

$$\sum_{k=0}^{\infty} \frac{w^k}{2^k},$$

which converges only if $2 > |w| = |z|^3$ by either the ratio or root tests. Hence we must have $|z| < 2^{1/3} = R$.

Problem 3: (4 points) Consider the complex valued function

$$f(z) = \frac{z^3 + 1}{(z - i)^3}.$$

(a) (1 point) Is the point $z_0 = i$ a removable singularity or a pole?

$$\lim_{z \to i} \left| \frac{z^3 + 1}{(z - 1)^3} \right| = \infty,$$

so $z_0 = i$ is a pole.

(b) (2 points) Find the Laurent series for f about $z_0 = i$.

$$f(z) = \frac{p(z)}{(z - i)^3},$$

where $p(z) = z^3 + 1$. We may rewrite $p(z) = \sum_{k=0}^{\infty} \frac{p^{(k)}(z_0)}{k!}(z - z_0)$ where $z_0 = i$ since p is entire as

$$p(z) = (1 - i) - 3(z - i) + 3i(z - i)^2 + 3(z - i)^3.$$

Hence

$$f(z) = \frac{1 - i}{(z - i)^3} - \frac{3}{(z - i)^2} + \frac{3i}{(z - i)} + 2,$$

is the Laurent series.

(c) (1 point) Calculate

$$\text{Res}(f; z_0) = \frac{1}{2\pi i} \int_{|\zeta - z_0| = s} f(\zeta) d\zeta.$$

We may simply read it off the answer to part (b) as

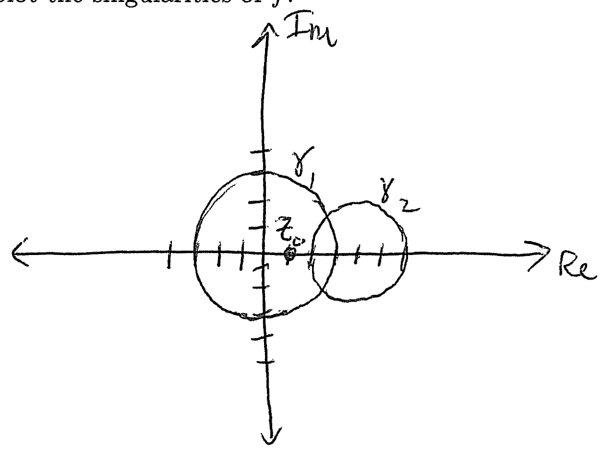
$$\text{Res}(f; z_0) = 3i.$$

Problem 4: (3 points) Consider the complex valued function

$$f(z) = \frac{z^3 + z^2}{(z - 1)^2}.$$

Let γ_1 be a parametrization of the circle $|z| = 3$ and γ_2 be a parametrization of the circle $|z - 4| = 2$.

(a) (1 point) Draw γ_1 and γ_2 and plot the singularities of f.



(b) (2 points) Calculate

$$\int_{\gamma_1} f(\zeta)d\zeta,$$

and

$$\int_{\gamma_2} f(\zeta)d\zeta.$$

(Hint, think about using the Residue Theorem.)

Using the Residue Theorem

$$\int_{\gamma_1} f(\zeta)d\zeta = 10\pi i$$

and

$$\int_{\gamma_2} f(\zeta)d\zeta = 0.$$