HOMEWORK 9 **DUE: Fri., Mar. 5** **NAME:**

DIRECTIONS:

- **STAPLE** this page to the front of your homework (don't forget your name!).
- Show all work, clearly and in order **You will lose points if you work is not in order**.
- When required, **do not forget the units**!
- Circle your final answers. **You will lose points if you do not circle your answers**.

Question	Points	Score
1	2	
2	1	
3	2	
4	2	
5	2	
6	1	
Total	10	

Problem 1: (2 points) Calculate the surface area of the section of the cone given by $z^2 = x^2 + y^2$ where $0 \leq z \leq 2$ using surface area integrals.

We begin by parametrizing the surface

$$\mathbf{\Psi}(u,v) = (u\cos v, u\sin v, u)\,,$$

for $u \in [0,2]$ and $v \in [0,2\pi]$. Then

$$\begin{aligned} \mathbf{\Psi}_u &= (\cos v, \sin v, 1)\,, \\ \mathbf{\Psi}_v &= (-u\sin v, u\cos v, 0)\,, \\ \mathbf{\Psi}_u \times \mathbf{\Psi}_\mathbf{v} &= (-u\cos v, -u\sin v, u)\,, \\ ||\mathbf{\Psi}_u \times \mathbf{\Psi}_\mathbf{v}|| &= \sqrt{2}u. \end{aligned}$$

So the surface area is given by

$$\int\int_D ||\mathbf{\Psi}_u \times \mathbf{\Psi}_\mathbf{v}||dA = \int_0^2 \int_-^{2\pi} \sqrt{2}udvdu = 4\pi\sqrt{2}.$$

Problem 2: (1 point) Is the surface discussed in problem 1, regular at all points (x_0, y_0, z_0) in its domain? Justify your answer.

We first note that

$$\mathbf{\Psi}_u \times \mathbf{\Psi}_v = (-u\cos v, -u\sin v, u) = \mathbf{0},$$

only when $u = 0$. Hence, the surface is not regular (i.e. smooth) at the cusp when $u = z = 0$.

Problem 3: (2 points) Suppose S is the graph of the portion of the paraboloid $z = 4 - x^2 - y^2$ where (x, y) varies throughout the disk $D = \left\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 4\right\}$. Calculate

$$\int\int_{\mathbf{\Psi}} (4 - z) dS,$$

where $\mathbf{\Psi}$ is a parametrization of the surface of the graph.

We may parametrize the surface with

$$\mathbf{\Psi} = \left(u, v, 4 - u^2 - v^2\right).$$

Hence

$$\begin{aligned} \mathbf{\Psi}_u &= (1, 0, -2u), \\ \mathbf{\Psi}_v &= (0, 1, -2v), \\ \mathbf{\Psi}_u \times \mathbf{\Psi}_\mathbf{v} &= (2u, 2v, 1), \\ ||\mathbf{\Psi}_u \times \mathbf{\Psi}_\mathbf{v}|| &= \sqrt{4u^2 + 4v^2 + 1}. \end{aligned}$$

Therefore

$$\int\int_{\mathbf{\Psi}} (4 - z) dS = \int_D \left(4 - \left[4 - u^2 - v^2\right]\right) \sqrt{4u^2 + 4v^2 + 1} du dv$$

If we switch to polar coordinates $dudv = rdrd\theta$ and $r \in [0, 2]$, $\theta \in [0, 2\pi]$ with $x = r\cos\theta$, $y = r\sin\theta$ we find

$$\int_0^2 \int_0^{2\pi} r^2 \sqrt{4r^2 + 1} r d\theta dr = 2\pi \int_0^2 r^3 \sqrt{4r^2 + 1} dr.$$

Letting $2r = tanw \Longrightarrow 2dr = \sec^2 w dw$, we find $r = 0 \to w = 0$ and $r = 2 \to w = tan^{-1}(4)$ so the integral becomes

$$\frac{\pi}{8} \int_0^{\tan^{-1}(4)} \tan^2 w \sec^2 w (\sec w \tan w) dw = \frac{\pi}{8} \int_0^{\tan^{-1}(4)} (\sec^2 w - 1) \sec^2 w (\sec w \tan w) dw.$$

using the identity $\tan^2 w = \sec^2 w - 1$. Now let $t = \sec w \Longrightarrow dt = \sec w \tan w$ with $w = 0 \to t = 1$ and $w = \tan^{-1}(4) \to t = \sqrt{17}$. Hence

$$\frac{\pi}{8} \int_1^{\sqrt{17}} t^2 \left(t^2 - 1\right) dt = \pi \left(\frac{391\sqrt{17} + 1}{60}\right).$$

Problem 4: (2 points) Find

$$\int\int_S x^2 dS,$$

where S is the surface of the cube $[-2, 2] \times [-2, -2] \times [-2, 2]$.

Parametrizing each face of the cube separately with $\mathbf{\Psi}_i : [-2,2] \times [-2,2] \to \mathbb{R}^3$ for $i \in \{1,2,3,4,5,6\}$ yields

$$\begin{aligned}
\mathbf{\Psi}_1 &= (-2, s, t), \\
\mathbf{\Psi}_2 &= (2, s, t), \\
\mathbf{\Psi}_3 &= (s, -2, t), \\
\mathbf{\Psi}_4 &= (s, 2, t), \\
\mathbf{\Psi}_5 &= (s, t, -2), \\
\mathbf{\Psi}_6 &= (s, t, 2).
\end{aligned}$$

Then the normals become $\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{i}$, $\mathbf{N}_3 = \mathbf{N}_4 = -\mathbf{j}$, and $\mathbf{N}_5 = \mathbf{N}_6 = \mathbf{k}$. Note, this means that $||\mathbf{N}_i|| = 1$ for all i. Hence

$$\int\int_S x^2 dS = \sum_{i=1}^{6} \left(\int_{\mathbf{\Psi}_i} x^2 ||\mathbf{N}_i|| dsdt \right).$$

By integrating each part separately, we find

$$\begin{aligned}
\int_{\mathbf{\Psi}_1} x^2 dsdt &= \int_{\mathbf{\Psi}_2} x^2 dsdt = \int_{-2}^{2}\int_{-2}^{2} 4dsdt = 64, \\
\int_{\mathbf{\Psi}_3} x^2 dsdt &= \int_{\mathbf{\Psi}_4} x^2 dsdt = \int_{-2}^{2}\int_{-2}^{2} s^2 dsdt = \frac{64}{3}, \\
\int_{\mathbf{\Psi}_5} x^2 dsdt &= \int_{\mathbf{\Psi}_6} x^2 dsdt = \int_{-2}^{2}\int_{-2}^{2} s^2 dsdt = \frac{64}{3}.
\end{aligned}$$

Hence

$$\int\int_S x^2 dS = \sum_{i=1}^{6} \left(\int_{\mathbf{\Psi}_i} x^2 ||\mathbf{N}_i|| dsdt \right) = \frac{640}{3}.$$

Problem 5: (2 points) Find the flux of the vector field $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} - \mathbf{k}$ across the upper hemisphere $x^2 + y^2 + z^2 = a^2$, where $z \geq 0$. Orient the hemisphere with an upward-pointing normal.

Parametrizing the surface yields

$$\mathbf{\Psi}(\phi, \theta) = (a\cos\theta\sin\phi, a\sin\theta\sin\phi, -a\cos\phi),$$

for $\phi \in [0, \pi/2]$, $\theta \in [0, 2\pi]$. Hence

$$\begin{aligned}
\mathbf{\Psi}_\phi &= (a\cos\theta\cos\phi, a\sin\theta\cos\phi, -a\sin\phi), \\
\mathbf{\Psi}_\theta &= (-a\sin\theta\sin\phi, a\cos\theta\sin\phi, 0), \\
\mathbf{N} = \mathbf{\Psi}_\phi \times \mathbf{\Psi}_\theta &= (a^2\cos\theta\sin^2\phi, a^2\sin\theta\sin^2\phi, a^2\sin\phi\cos\phi), \\
||\mathbf{\Psi}_\phi \times \mathbf{\Psi}_\theta|| &= a^2\sin\phi, \\
\mathbf{n} = \mathbf{N}/||\mathbf{N}|| &= (\cos\theta, \sin\theta, \cot\phi).
\end{aligned}$$

We see that $\mathbf{n}$ is, in fact, the outward facing normal so the flux is given by

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} = \int\int_{\mathbf{\Psi}} \mathbf{F} \cdot \mathbf{N} d\theta d\phi = \int_0^{\pi/2}\int_0^{2\pi} -a^2\sin\phi\cos\phi d\theta d\phi = -2\pi a^2 \int_0^{\pi/2} \sin\phi\cos\phi d\phi.$$

Letting $u = \sin\phi \Longrightarrow du = \cos\phi d\phi$ with $\phi = 0 \to u = 1$ and $\phi = \pi/2 \to u = 1$, so the integral becomes

$$-2\pi a^2 \int_0^1 udu = -\pi a^2.$$

Problem 6: (1 point) Find the Gauss curvature of the hyperbolic paraboloid

$$z = \frac{x^2}{a^2} - \frac{y^2}{b^2}.$$

If $\mathbf{\Psi}(u,v) = \left(u, v, \frac{u^2}{a^2} - \frac{v^2}{b^2}\right)$ is the parametrization of the hyperbolic paraboloid, the Gauss curvature is given by

$$K = \frac{ln - m^2}{W},$$

where

$$\begin{aligned} l &= \mathbf{N} \cdot \mathbf{\Psi}_{uu}, \\ n &= \mathbf{N} \cdot \mathbf{\Psi}_{vv}, \\ m &= \mathbf{N} \cdot \mathbf{\Psi}_{uv}, \\ W &= ||\mathbf{\Psi}_u \times \mathbf{\Psi}_v||^2. \end{aligned}$$

Calculating these shows

$$\begin{aligned} l &= \frac{2}{a^2\sqrt{W}}, \\ n &= \frac{-2}{b^2\sqrt{W}}, \\ m &= 0, \\ W &= 1 + \frac{4u^2}{a^4} + \frac{4v^2}{b^4}. \end{aligned}$$

Hence

$$K = \frac{ln - m^2}{W} = \frac{-4a^6b^6}{\left(a^4b^4 + 4b^4u^2 + 4a^4v^2\right)^2}.$$