REVIEW SHEET FOR EXAM II **Friday, May 14**

DIRECTIONS:

- This exam will be closed book, closed notes.
- NO CALCULATORS allowed.
- Show all work, clearly and in order.
- Circle your final answers. **You will lose points if you do not circle your answers**.
- This exam will have between 5 and 7 problems, with 1 extra credit problem.

Problem Type 1: Determine the form of the general solution of the following differential equations.

$$\begin{aligned} y''' - 2y'' + y' &= x^3 + 2e^x, \\ y^{(4)} - 2y'' + y &= e^x + \sin x, \\ y^{(4)} + y''' &= \sin 2x. \end{aligned}$$

(Hint: Considering writing the equation using differential operators and think about annihilators.)

Problem Type 2: Use variation of parameters to determine the general solution for the following differential equations

$$\begin{aligned} y''' - y'' - y' + y &= e^x, \\ y''' + y' &= \sec x, \\ y^{(4)} + 2y'' + y &= \sin x. \end{aligned}$$

Problem Type 3: Determine the radii and intervals of convergence for the following series.

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n2^n},$$

$$\sum_{k=0}^{\infty} x^k,$$

$$\sum_{k=0}^{\infty} \frac{(-1)^{2k}}{(2k)!} x^{2k}.$$

Problem Type 4: Find and classify the singular points of the following differential equations. Solve the differential equations using series solutions about the ordinary point $x_0 = 0$.

$$\begin{aligned} y'' - xy &= 0, \\ (4 - x^2)y'' + 2y &= 0. \end{aligned}$$

Problem Type 5: Find one series solution to the following differential equations about the regular singular point $x_0 = 0$. What can you say about the certainty of getting two linearly independent solutions if you were to apply Frobenius's Method?

$$\begin{aligned} 2xy'' + y' + xy &= 0, \\ x^2y'' + xy' + (x-2)y &= 0, \\ x^2y'' - x(x+3)y' + (x+3)y &= 0. \end{aligned}$$

Problem Type 6: Consider the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

WITHOUT solving, what can you say about the existence of series solution(s) about the point $x_0 = 0$? If solutions exist, what will their radius of convergence be?

Problem Type 7: For example problems on complex analysis, review your class notes and the homework problems from HW7.

Note: Just because a formula or theorem is given here, does not mean that it is necessary for any of the given problems. Use these as needed only.

FORMULAE:

$$
\begin{aligned}
e^{i\theta} &= \cos\theta + i\sin\theta \\
e^x &= \textstyle\sum_{k=0}^{\infty} \frac{x^k}{k!} \quad \text{for all x} \\
\log(1-x) &= -\textstyle\sum_{k=1}^{\infty} \frac{x^k}{k} \quad \text{for } x \in [-1,1) \\
\tfrac{1}{1-x} &= \textstyle\sum_{k=0}^{\infty} x^k \quad \text{for } x \in (-1,1) \\
\tfrac{x}{(1-x)^2} &= \textstyle\sum_{k=1}^{\infty} kx^k \quad \text{for } x \in (-1,1) \\
\sin x &= \textstyle\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \text{for all x} \\
\cos x &= \textstyle\sum_{k=0}^{\infty} \frac{(-1)^{2k}}{(2k)!} x^{2k} \quad \text{for all x}
\end{aligned}
$$

A few useful annihilators are given by

$$
\begin{aligned}
D^n[c_0 + c_1 x + \cdots c_{n-1}x^{n-1}] &= 0, \\
(D-\alpha)^n\left[\left(c_0 + c_1 x + \cdots c_{n-1}x^{n-1}\right)e^{\alpha x}\right] &= 0.
\end{aligned}
$$

The radius of convergence of the series $\sum_{n=0}^{\infty} c_n x^n$ is given by

$$
R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|.
$$

If $z^n = w$ then the roots of this equation are given by

$$
\begin{aligned}
z_k &= |w|^{1/n}\left(\cos\theta_k + i\sin\theta_k\right), \\
\theta_k &= \frac{\mathrm{Arg} w}{n} + k\left(\frac{2\pi}{n}\right),
\end{aligned}
$$

where $k = 0, 1, 2, ..., n-1$.

THEOREM 1: Consider the first order, initial value problem

$$
y' = f(x,y), \quad y(x_0) = y_0,
$$

and a rectangle, R, in the xy-plane such that $(x_0, y_0) \in R$. If f and $\frac{\partial f}{\partial y}$ are continuous on R, then there exists an interval, I, centered at x_0, and a unique solution $y(x)$ on I such that y satisfies the above initial value problem.

THEOREM 2: Consider the second order, linear, initial value problem

$$
y'' + p(x)y' + q(x)y = g(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_0',
$$

where p, q, and g are continuous on an open interval, I, such that $x_0 \in I$. Then there exists a unique solution $y(x)$ on I such that y satisfies the above initial value problem.

THEOREM 3: Consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

If $x = x_0$ is an ordinary point of this equation, then we can find two linearly independent solutions of the form

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

which will converged for $|x - x_0| < R$, where R is the distance from x_0 to the nearest singular point, real or complex.

THEOREM 4: (Frobenius' Theorem) If $x = x_0$ is a regular singular point of the differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0,$$

then there exists at least one series solution of the form

$$y = \sum_{n=0}^{\infty} c_n \left(x - x_0\right)^{n+r},$$

where r is a constant that must be determined (one of the indicial roots) and the series will converge at least on some interval $|x - x_0| < R$.