FINAL EXAM **THURSDAY, JUNE 10** **NAME:**

DIRECTIONS:

- This exam will be closed book, closed notes.
- NO CALCULATORS allowed.
- Show all work, clearly and in order.
- Circle your final answers. **You will lose points if you do not circle your answers**.
- This exam will have between 9 and 11 problems, with 1 extra credit problem.

Problem Type 1: Be familiar with complex trigonometric functions. That is, be able to prove the following

$$\begin{aligned}
\cos z &= \cos x \cosh y - i \sin x \sinh y \\
\sin z &= \sin x \cosh y + i \cos x \sinh y \\
\cos(z+w) &= \cos z \cos w - \sin z \sin w \\
\sin(z+w) &= \sin z \cos w + \cos z \sin w \\
\text{Arcsin} z &= -i \text{Log}\left(iz + \sqrt{1-z^2}\right) \\
\text{Arccos} z &= -i \text{Log}\left(z + \sqrt{z^2-1}\right) \\
\text{Arctan} z &= \frac{i}{2} \text{Log}\left(\frac{1-iz}{1+iz}\right)
\end{aligned}$$

where $z = x + iy \in \mathbb{C}$ and $x, y \in \mathbb{R}$. Note $\text{Arcsin}z = \sin^{-1} z$.

Problem Type 2: Be able to write z^w where $z, w \in \mathbb{C}$ as a number you can plot in the complex plane (i.e. as $x + iy$ where $x, y \in \mathbb{R}$. For example

(a) i^{1+i},

(b) π^i.

Problem Type 3: In your own words, be able to define or describe:

(a) Suppose D is a domain. What does this mean?

(b) What does it mean for a set to be open?

(c) What does it mean for a set to be connected?

(d) What does it mean for a set to be closed?

(e) What does it mean for a set to be simply connected?

(f) What does it mean for a complex valued function, f, to be analytic at a point z_0 in a domain D?

(g) What does it mean for f to be harmonic?

(h) How does analycity for $f(z)$ differ from continuity for a real valued function, $g(x)$?

(i) Why do we learn complex analysis in a differential equations class?

(j) Describe how you might use differential equations, mathematical modeling, or complex analysis (pick one) in real life.

Problem Type 4: Be able to state and use the Cauchy-Riemann equations. For example, suppose $u(x,y) = x^2 - y^2$. Find $f(x,y) = u(x,y) + iv(x,y)$ such that f is analytic.

Problem Type 5: Consider the function

$$f(z) = \frac{z^2 + z^4}{(z-2)^2}.$$

(a) Identify and classify the singularities (i.e. is $z_0 = 2$ a removable singularity or a pole? What is the difference?)

(a) Calculate the Laurent series of f about it's singularity.

(c) What is the residue of f at the singularity?

(d) Let γ_1 be a parametrization of the circle $|z| = 3$ and γ_2 be a parametrization of the circle $|z - 10| = 2$. Draw γ_1 and γ_2 and plot the singularities of f.

(e) Calculate

$$\int_{\gamma_i} f(\zeta) d\zeta.$$

Problem Type 6: You will be asked to verify the Laplace Transform for one of the following

$f(t)$	$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$\delta(t-c)$	e^{-cs}
$u_c f(t-c)$	$e^{-cs} F(s)$

Problem Type 7: Solve the initial value problem

$$ay'' + by' + cy = g(t), \ y(0) = y_0, \ y'(0) = y_0',$$

using Laplace Transforms. For example

(a) $y'' + 3y' + 2y = 0$ where $y(0) = 1$ and $y'(0) = 1$.

(b) $y'' + y = \sin 2t$ where $y(0) = 2$ and $y'(0) = 1$. (Note, this one is an example in your book in section 6.2).

Note: Just because a formula or theorem is given here, does not mean that it is necessary for any of the given problems. Use these as needed only.

LAPLACE TRANSFORMS:

$f(t)$	$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
t^n	$\frac{n!}{s^{n+1}}$, n a positive integer
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$u_c(t)$	$\frac{e^{-cs}}{s}$
$\delta(t-c)$	e^{-cs}
$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$

FORMULAE:

If $z^n = w$ then the roots of this equation are given by

$$\begin{aligned} z_k &= |w|^{1/n} \left(\cos\theta_k + i \sin\theta_k \right), \\ \theta_k &= \frac{\text{Arg}w}{n} + k \left(\frac{2\pi}{n} \right), \end{aligned}$$

where $k = 0, 1, 2, ..., n-1$.
The step, or Heaviside function is given by

$$u_c(t) = \begin{cases} 0, & t < c \\ 1, & t \geq c \end{cases}$$

$$
\begin{aligned}
\log z &= \ln|z| + i\arg(z) \\
\text{Log}z &= \ln|z| + i\text{Arg}(z) \\
e^{ix} &= \cos x + i\sin x \\
\sin z &= \tfrac{1}{2i}\left(e^{iz} - e^{-iz}\right) \\
\cos z &= \tfrac{1}{2}\left(e^{iz} + e^{-iz}\right) \\
\tan z &= \tfrac{\sin z}{\cos z} \\
\sinh x &= \tfrac{1}{2}\left(e^{x} - e^{-x}\right) \\
\cosh x &= \tfrac{1}{2}\left(e^{x} + e^{-x}\right) \\
\text{Arcsin}z &= -i\text{Log}\left(iz + \sqrt{1-z^2}\right) \\
\text{Arccos}z &= -i\text{Log}\left(z + \sqrt{z^2-1}\right) \\
\text{Arctan}z &= \tfrac{i}{2}\text{Log}\left(\tfrac{1-iz}{1+iz}\right)
\end{aligned}
$$

where $z \in \mathbb{C}$ and $x \in \mathbb{R}$.

THEOREM 1: Consider the equation

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

If $x = x_0$ is an ordinary point of this equation, then we can find two linearly independent solutions of the form

$$y = \sum_{n=0}^{\infty} c_n(x-x_0)^n,$$

which will converged for $|x - x_0| < R$, where R is the distance from x_0 to the nearest singular point, real or complex.

THEOREM 2: (Cauchy-Riemann equations) Consider the complex valued function $f = u + iv$. If f is analytic on a domain D, then f satisfies the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

THEOREM 3: (Cauchy's Theorem) Suppose f is analytic on a domain D. Let γ be a piecewise smooth, simple, closed curve in D whose inside, Ω, is also in D. Then

$$\int_\gamma f(z)dz = 0.$$

THEOREM 4: (Residue Theorem) Suppose f is analytic on a simply-connected domain D except at a finite number of isolated singularities at $z_1, z_2, ..., z_N$ of D. Let γ be a piecewise smooth, positively oriented, simple closed curve in D that does not pass through $z_1, z_2, ..., z_N$. Then

$$\int_\gamma f(z)dz = 2\pi i \sum_{z_k \text{ inside } \gamma} \text{Res}(f; z_k).$$