HOMEWORK 2 **DUE: Fri., Apr. 6** **NAME:**

DIRECTIONS:

- Turn in your homework as **SINGLE-SIDED** typed or handwritten pages.
- **STAPLE** your homework together. Do not use paper clips, folds, etc.
- **STAPLE** this page to the front of your homework.
- Be sure to write your name on your homework.
- Show all work, **clearly and in order**.

You will lose point 0.5 points for each instruction not followed.

Questions	Points	Score
1	1	
2	1	
3	1	
4	2	
5	3	
6	1	
7	1	
Total	10	

Problem 1: (1 point) Suppose $A \neq \emptyset$ and $B \neq \emptyset$. Show that $A \times B = B \times A$ iff $A = B$.

Proof: Assume $A \neq \emptyset$ and $B \neq \emptyset$.

Part 1: Assume $A \times B = B \times A$. By definition of equality of sets, this means that every element of $A \times B$ is an element of $B \times A$ so there exist elements $p, m \in A$ and elements $q, l \in B$ such that $x = (p, q) \in A \times B$ and $x = (l, m) \in B \times A$ and $(p, q) = (l, m)$. But by the definition of ordered pairs, that means $p = l$ and $q = m$ so for all $p \in A$, there exists an $l \in B$ such that $p = l$ hence $A \subset B$. Similarly, $B \subset A$.

Part 2: Assume $A = B$. Let $(p, q) \in A \times B$. Hence $p \in A = B \Longrightarrow p \in B$ and $q \in B = A \Longrightarrow q \in A$. Hence $(p, q) \in B \times A$ as well by definition of Cartesian product, hence $A \times B \subset of B \times A$. Similarly, $B \times A \subset A \times B$.

Parts 1 and 2 imply that assuming $A \neq \emptyset$ and $B \neq \emptyset$ then $A \times B = B \times A$ iff $A = B$. Q. E. D.

Problem 2: (1 point) If A, B, and C are finite sets, show that

$$^{\#}(A \cup B \cup C) =^{\#} A +^{\#} B +^{\#} C -^{\#} (A \cap B) -^{\#} (A \cap C) -^{\#} (B \cap C) +^{\#} (A \cap B \cap C).$$

Proof: Since this is equality of numbers, rather than sets, it suffices to show that for element x in the universal set X contributes the same number to both sides of the equation above. An element $x \in A \cup B \cup C$ or $x \notin A \cup B \cup C$.

Part 1: $x \notin A \cup B \cup C \iff x \notin A$ and $x \notin B$ and $x \notin C$. So x is also not in any of the intersections of two or three of these sets. For such x, the contributions are

$$\begin{aligned} L.H.S. &= +0, \\ R.H.S. &= +0. \end{aligned}$$

So the element contributes exactly zero to both sides.

Part 2 $x \in A \cup B \cup C \iff$

1. x is in exactly one of the sets (e.g. $x \in A$ but $x \notin B$ and $x \notin C$)

$$\begin{aligned} L.H.S. &= +1, \\ R.H.S. &= +1 + 0 + 0 - 0 - 0 - 0 + 0. \end{aligned}$$

2. x is in exactly two of the sets (e.g. $x \in A$ and $x \in B$ but $x \notin C$)

$$\begin{aligned} L.H.S. &= +1, \\ R.H.S. &= +1 + 1 + 0 - 1 - 0 - 0 + 0. \end{aligned}$$

3. x is in exactly three of the sets (i.e. $x \in A$, $x \in B$, and $x \in C$

$$\begin{aligned} L.H.S. &= +1, \\ R.H.S. &= +1 + 1 + 1 - 1 - 1 - 1 + 1. \end{aligned}$$

In any case, the element contributes exactly one to both sides of the equation.

Q.E.D.

Problem 3: (1 point) If $a, b \in \mathbb{Z}$, show $(-a)(-b) = ab$.

Proof: Let a, $b \in \mathbb{Z}$. By convention we know that $-a = (-1)a$ and $-b = (-1)b$. So

$$(-a)(-b) = (-1)(a)(-1)(b) = (-1)(-1)(ab),$$

by associativity. But we also know that -1 is its own multiplicative inverse so $(-1)(-1) = 1$, hence

$$(-a)(-b) = (1)ab = ab,$$

since 1 is the multiplicative identity.

Q. E. D.

Problem 4: (2 points) If a, $b \in \mathbb{Z}$,

(a) (1 point) Suppose $0 < a$ and $0 < b$. Show that $a < b$ iff $a^2 < b^2$.

Proof: Suppose $0 < a$ and $0 < b$.

Part I: Assume $a < b$, then since a and b are positive we may multiply both sides by a and b to obtain $a^2 < ab$ and $ab < b^2$. Since $<$ is transitive, $a^2 < b^2$.

Part II: Assume $a^2 < b^2$. Hint, suppose we do not have $a < b$, so either $a = b$ or $b < a$. By the same approach as in part I, we arrive at a contradiction for both options.

Q.E.D.

(b) (1 point) Suppose $a < 0$ and $b < 0$. Show that $a < b$ iff $b^2 < a^2$.

Proof: Suppose $a < 0$ and $b < 0$.

Part I: Assume $a < b$, then $-b < -a$ from facts 1.5.5 in the text. By the same proof as (a) we are done.

Part II: Follow the same procedure as in part (a).

Problem 5: (3 points) If n, k are non-negative integers, we define the binomial coefficient, $\binom{n}{k}$, by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!},$$

where $n! = n \cdot (n-1) \cdots 2 \cdot 1$, and we set $0! = 1$.

(a) (2 point) Prove that

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r},$$

for $r = 1, 2, 3, \ldots$, n

Proof: We will use proof by induction. Consider the case when $r = 1$,

$$\binom{n}{1} + \binom{n}{0} = n + 1 = \binom{n+1}{1}.$$

Now let us assume that for some $r = k$

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k},$$

which is the same as writing

$$\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!} = \frac{(n+1)!}{k!(n+1-k)!}.$$

We want to show that

$$\binom{n}{k+1} + \binom{n}{k} = \binom{n+1}{k+1},$$

which is the same as writing

$$\frac{n!}{(k+1)k!(n-k-1)!} + \frac{n!}{k!(n-k)!} = \frac{(n+1)!}{(k+1)!(n-k)!}.$$

We notice that if we multiply the equation for $r = k$ by $\frac{m+1-k}{k+1}$ on both sides, the R.H.S. becomes exactly what we want in the equation for $r = k+1$. That is

$$\frac{m+1-k}{k+1}\left[\frac{n!}{k!(n-k)!} + \frac{n!}{(k-1)!(n-k+1)!}\right] = \frac{(n+1)!}{(k+1)!(n-k)!}.$$

With a little algebra, we see that the L.H.S. is exactly what we want for $r = k+1$.

Q.E.D.

(b) (1 points) Using part (a), prove the Binomial Theorem:

If a, $b \in \mathbb{Z}$ and n is a positive integer, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Hint: Use mathematical induction

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Hint: Use mathematical induction

Proof: We will use proof by induction. Consider the case when $n = 1$,

$$(a+b)^1 = a+b,$$

and

$$\sum_{k=0}^{1} \binom{1}{k} a^k b^{1-k} = \binom{1}{0} b + \binom{1}{1} a = a+b.$$

Now let us assume that this holds true for some $n = m$, that is

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k},$$

and we W.T.S. that

$$(a+b)^{m+1} = \sum_{k=0}^{m+1} \binom{m+1}{k} a^k b^{m+1-k}.$$

We first notice that the L.H.S. for the $m+1$ case is precisely $(a+b)$ times the L.H.S. for the m case. So, let us multiply by $(a+b)$

$$\begin{aligned}
(a+b)^{m+1} &= \textstyle\sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k}(a+b) \\
&= \textstyle\sum_{k=0}^{m} \binom{m}{k} \left[a^{k+1}b^{m-k} + a^k b^{m+1-k}\right] \\
&= \binom{m}{0}\left[ab^m + b^{m+1}\right] + \binom{m}{1}\left[a^2b^{m-1} + ab^m\right] + \binom{m}{2}\left[a^3b^{m-1} + a^2b^{m-1}\right] + \binom{m}{3}\left[a^4b^{m-3} + a^3b^{m-2}\right] \\
&\quad + \cdots + \binom{m}{m-1}\left[a^m b + a^{m-1}b^2\right] + \binom{m}{m}\left[a^{m+1} + a^m b\right] \\
&= \binom{m}{0} b^{m+1} + \left[\binom{m}{0} + \binom{m}{1}\right] ab^m + \left[\binom{m}{1} + \binom{m}{2}\right] a^2 b^{m-1} + \cdots + \left[\binom{m}{m-1} + \binom{m}{m}\right] a^m b + \binom{m}{m} a^{m+1} \\
&= \textstyle\sum_{k=0}^{m+1} \binom{m+1}{k} a^k b^{m+1-k},
\end{aligned}$$

by using part (a).

Q.E.D.

Problem 6: (1 point) Let n be an integer greater than or equal to 2. If $a, b \in \mathbb{Z}$, we say that $a \sim b$ iff $a - b$ is a multiple of n, that is, n divides $a - b$. Prove this defines an equivalence relation.

Proof: It suffices to show that this relation is (1) reflexive, (2) symmetric, and (3) transitive.

1. *Reflexive;* $(a - a) = 0 = 0 \cdot n$ so $a \sim a$.
2. *Symmetric:* Suppose $a \sim b$ then there exists $c \in mathbbZ$ such that $a - b = cn$ hence $-(b - a) = cn \Longrightarrow b - a = (-c)n$. So $b \sim a$.
3. *Transitive:* Suppose $a \sim b$, and $b \sim c$, then there exist $e, g \in \mathbb{Z}$ such that $a - b = en$ and $b - c = fn$. Hence $(a - b) - (b - c) = en - fn \Longrightarrow a - c = (e - f)$ where $(e - f) \in \mathbb{Z}$.

Q.E.D.

Problem 7: (1 point) Let n be a positive integer greater than or equal to 2. Then there exists a prime p such that p divides n.

Hint: Consider using the Principle of Strong Induction: To prove an infinite sequence of statements $p(n)$ for $n = b, b+1, ...,$ prove the following implication for $k = b, b+1, b+2, ...$: $p(m)$ for all m such that $b \le m < k \Longrightarrow p(k)$.

Proof: Consider $n = 2$, then $2|2$ and 2 is prime. Now suppose every integer less than k but greater than or equal to 2 has a prime factor. If k prime then we are done. If not, then k is composite and $k = cm$ for c, $m \in [2, 3, 4, ..., k-1] \Longrightarrow m$ has a prime factor, hence $m = dp$. So $k = (cd)p$ so $p|k$.

Q.E.D.