HOMEWORK 3 **DUE: Fri., Apr. 13** **NAME:**

DIRECTIONS:

- Turn in your homework as **SINGLE-SIDED** typed or handwritten pages.
- **STAPLE** your homework together. Do not use paper clips, folds, etc.
- **STAPLE** this page to the front of your homework.
- Be sure to write your name on your homework.
- Show all work, **clearly and in order**.

You will lose point 0.5 points if one or more of these instructions are not followed.

Questions	Points	Score
1	1	
2	1	
3	1	
4	1	
5	1	
Total	5	

Problem 1: (1 point)

(a) (0.5 point) Show that $\{(-a, b)\}$ is an additive inverse for $\{(a, b)\}$.

Proof: Consider $\{(a, b)\}+\{(-a, b)\} = \{(ab+b(-a), b^2)\} = \{(0, b^2)\} = \{(0, 1)\}$, which is the additive identity in $\mathbb{Q}$. Q.E.D.

(b) (0.5 point) Prove the distributive law for $\mathbb{Q}$.

Proof: Let $\{(a, b)\}$, $\{(c, d)\}$, and $\{(e, g)\} \in \mathbb{Q}$. We want to show

$$\{(a, b)\} \cdot [\{(a, b)\} + \{(e, g)\}] = \{(a, b)\} \cdot \{(c, d)\} + \{(a, b)\} \cdot \{(e, f)\}.$$

A little algebra shows that

$$\begin{aligned} L.H.S. &= \{(a, b)\} \cdot \{(cf + de, df)\} = \{(acf + ade, bdf)\}, \\ R.H.S &= \{(ac, bd)\} + \{(ae, bf)\} = \{(acbf + bdae, b^2df)\} = \{(b, b)\} \cdot \{(acf + dae, bdf)\}, \end{aligned}$$

but $\{(b, b)\} = \{(1, 1)\}$ which is the multiplicative identity in $\mathbb{Q}$ so Q.E.D.

Problem 2: (1 points) Let R be a ring and R_0 a nonempty subset of R. Show that R_0 is a subring iff, for any a, $b \in R_0$, we have $a - b$, $ab \in R_0$.

Proof: Let R be a ring and $R_0 \subset R$ with $R \neq \emptyset$.

Part I: Suppose R_0 is a subring. Then R_0 is also closed under addition and multiplication so for all $a, b \in R_0 \Longrightarrow a \cdot b \in R_0$. Also, since R_0 is a ring, for all $b \in R_0$ it's additive inverse $-b \in R_0$ as well. Hence for all $a, b \in R_0 \Longrightarrow a + (-b) = a - b \in R_0$.

Part II: Suppose for all $a, b \in R_0$ then $a \cdot b \in R_0$ and $a - b \in R_0$. Then

- The first part tells us that R_0 is closed under multiplication.
- The fact that R is a ring tells us that R_0 is associative under addition and multiplication, and commutative under addtion (and *if* $\cdot$ is commutative in R then it is also commutative in R_0, and that the distributive rule(s) holds.

We want to show that

- $0 \in R_0$. We know for all $a \in R_0 \Longrightarrow 0 = a - a \in R_0$, so $0 \in R_0$.
- for all $a \in R_0$ we also have $-a \in R_0$. By the first part we know $a, 0 \in R_0$ so $-a = 0 - a \in R_0$, and we are done.
- it is closed under $+$. Suppose R_0 is not closed. Then there exist $a, b \in R_0$ such that $a + b \notin R_0$, but $a + b = a - (-b) \in R_0$ soR_0 must be closed under addition.

Q.E.D.

Problem 3: (1 points) Let X be a non-empty set and R be the power set of X. Prove that R with symmetric difference as addition and intersection as multiplication is a commutative ring with identity.

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See problem 1.3.9 in the book. You have already proven almost everything that is required here.

Problem 4: (1 point) Let $A = \{p, q, r\}$ and $B = \{\pi, e\}$. Determine all possible functions from A to B.

The functions are

1. $\{(p, \pi), (q, \pi), (r, \pi)\}$
2. $\{(p, e), (q, e), (r, e)\}$
3. $\{(p, \pi), (q, \pi), (r, e)\}$
4. $\{(p, \pi), (q, e), (r, \pi)\}$
5. $\{(p, e), (q, \pi), (r, \pi)\}$
6. $\{(p, \pi), (q, e), (r, e)\}$
7. $\{(p, e), (q, e), (r, \pi)\}$
8. $\{(p, e), (q, \pi), (r, e)\}$

Problem 5: (1 points) Given $f : A \to B$, suppose there exist $g, h : B \to A$ so that $f \circ g = I_B$ and $h \circ f = I_A$. Show that f is a bijection and that $g = h = f^{-1}$.

Proof: Let $f : A \to B$, and suppose there exist $g, h : B \to A$ so that $f \circ g = I_B$ and $h \circ f = I_A$.

Part I: We want to show that f is a bijection. First we will show onto. Let $b \in B$, then $(f \circ g)(b) = f(g(b)) = b$ call$g(b) = a \in A$ so there exists an $a \in A$ such that $f(a) = b$ for all $b \in B$. Hence f is onto. Now suppose $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$. Then

$$\begin{aligned} h(f(a_1) &= a_1, \\ h(f(a_2) &= a_2, \end{aligned}$$

but $b = f(a_1) = f(a_2)$ so $a_1 = h(b) = h(b) = a_2$ so f is 1-1. Hence f is a bijection.

Part II: We want to show that $g = h = f^{-a}$. Let $b \in B$ and $a \in A$. We know $f(g(b)) = b$ and $h(f(a)) = a$. So

$$h(b) = h(f(g(b))) = g(b).$$

Q.E.D.