HOMEWORK 6 **DUE: Mon., May 7** **NAME:**

DIRECTIONS:

- Turn in your homework as **SINGLE-SIDED** typed or handwritten pages.
- **STAPLE** your homework together. Do not use paper clips, folds, etc.
- **STAPLE** this page to the front of your homework.
- Be sure to write your name on your homework.
- Show all work, **clearly and in order**.

You will lose point 0.5 points for each instruction not followed.

Questions	Points	Score
1	2	
2	2	
3	1	
4	2	
5	2	
6	1	
Total	10	

Problem 1: (2 point) Let $x \in \mathbb{R}$, Show that there exists a sequence s such that $s_k \in \mathbb{Q}$ for all $k \in \mathbb{N}$ and $\lim_{n\to\infty} s_k = x$.

Proof: Let $k \in \mathbb{N}$. Then the interval $\left(x - \frac{1}{k}, x + \frac{1}{k}\right)$ contains a rational number we will call s_k. We choose s_k using the Axiom of Choice.

Let $\epsilon > 0$ be given. Then by the ϵ-property, there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. So

$$s_n \in \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \Longrightarrow x - \frac{1}{n} < s_n < x + \frac{1}{n},$$

hence

$$-\frac{1}{n} < s_n - x < \frac{1}{n} \Longrightarrow |s_n - x| < \frac{1}{n} < \epsilon.$$

Q.E.D.

Problem 2: (2 point) Show that for any a, $b \in \mathbb{Q}$, we have $||a| - |b|| \le |a - b|$.

Proof: We will assume that the triangle inequality for the rational numbers holds. That is, for all $x, y \in \mathbb{Q}$

$$|x + y| \le |x| + |y|.$$

First let $x = a - b$ and $y = b$. Then the triangle inequality yields

$$|a - b + b| \le |a - b| + |b| \Longrightarrow |a| - |b| \le |a - b|.$$

Now let $x = a$ and $y = b - a$. Then the triangle inequality yields

$$|a + b - a| \le |a| + |b - a| \Longrightarrow -(|a| - |b|) \le |a - b|.$$

Hence

$$||a| - |b|| \le |a - b|.$$

Q.E.D.

Problem 3: (2 points) Show that the sum of two Cauchy sequence in $\mathbb{Q}$ is a Cauchy sequence in $\mathbb{Q}$.

Proof: Let $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ be Cauchy sequences. We want to show that given $r \in \mathbb{Q}^+$, there exists an N such that for all $n > N$

$$|a_n + b_n - (a_m + b_m)| < r.$$

Let $r \in \mathbb{Q}^+$ be given. We know that since $(a_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ are Cauchy sequences, there exist $N_1, N_2 \in \mathbb{N}$ such that for all $n, m > N_1$

$$|a_n - a_m| < \frac{r}{2},$$

and for all $k, l > N_2$

$$|b_k - b_l| < \frac{r}{2}.$$

So choose $N = \max\{N_1, N_2\}$ so for all $n, m > N$ we have

$$|a_n + b_n - (a_m + b_m)| = |a_n - a_m + b_n - b_m| \le |a_n - a_m| + |b_n - b_m| < \frac{r}{2} + \frac{r}{2} = r.$$

Q.E.D.

Problem 4: (2 points) Show that addition is well defined in $\mathbb{R}$.

Proof: Let $(a_k)_{k\in\mathbb{N}}$, $(a'_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, and $(b'_k)_{k\in\mathbb{N}}$ be Cauchy sequences such that $(a_k)_{k\in\mathbb{N}} \sim (a'_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}} \sim (b'_k)_{k\in\mathbb{N}}$.

We want to show that given $(a_k + b_k)_{k\in\mathbb{N}} \sim (a'_k + b'_k)_{k\in\mathbb{N}}$. That is given $r \in \mathbb{Q}^+$, there exists an $N \in \mathbb{N}$ such that for all $n > N$ we have

$$|a_n + b_n - (a'_n + b'_n)\,| < r.$$

Let $r \in \mathbb{Q}^+$ be given. Then we know that there exist N_1 and N_2 such that if we choose $N = \max\{N_1, N_2\}$ then for all $n > N$ we have

$$|a_n - a'_n| < \frac{r}{2},$$

and

$$|b_n - b'_n| < \frac{r}{2}.$$

Hence

$$|a_n + b_n - (a'_n + b'_n)\,| \le |a_n - a'_n| + |b_n - b'_n| < \frac{r}{2} + \frac{r}{2} = r.$$

Q.E.D.

Problem 5: (2 points) Let $(a_k)_{k\in\mathbb{N}}$ be a Cauchy sequence of rational numbers such that $(a_k)_{k\in\mathbb{N}} \notin \mathcal{I}$. Define the inverse sequence, $(b_k)_{k\in\mathbb{N}}$, by

$$b_k = \begin{cases} 1, & \text{for } k \le N, \\ 1/a_k, & \text{for } k > N, \end{cases}$$

where for $n > N$ we know there exists an $r \in \mathbb{Q}^+$ such that $|a_k| > r$. Prove that $(b_k)_{k\in\mathbb{N}}$ is Cauchy.

Proof: From class we know there exists an $N_1 \in \mathbb{N}$ and $r \in \mathbb{R}$ such that for all $n > N_n$ we have

$$|a_n| > r,$$

So for $n, m > N_1$ we know

$$|b_n - b_m| = |1/a_n - 1/a_m| = |\frac{a_m - a_n}{a_n a_m}| < \frac{1}{r^2}|a_n - a_m|.$$

But $(a_m)_{m\in\mathbb{N}}$ is Cauchy so there exists an $N_2 \in \mathbb{N}$ such that for $m, n > N_2$ we know

$$|a_n - a_m| < r^3.$$

So take $m, n > N = \max\{N_1, N_2\}$ so

$$|b_n - b_m| < \frac{r^3}{r^2} = r.$$

Q. E. D.

Problem 6: (1 point) Prove that every bounded sequence in $\mathbb{R}$ has a convergent subsequence.

Proof: Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathbb{R}$. The we know it has a monotonic subsequence. Suppose that subsequence is monotonic increasing. That is there exists a subsequence $(b_{k_n})_{n\in\mathbb{N}}$ such that $b_{k_n} \le b_{k_m}$ for $m > n$. Define

$$A = \{b_{k_n} | n \in \mathbb{N}\} \subseteq \mathbb{R},$$

which is bounded above which implies that it has a least upper bound in $\mathbb{R}$, which we will call $u = lub(A)$. Let $\epsilon > 0$ be given. Then $u - \epsilon < u$ is not an upper bound, so for some n

$$u - \epsilon < b_{k_n} \leq b_{k_m} < u,$$

for all $m > n$ since this is a monotonic increasing subsequence. So

$$|b_{k_m} - u| = b_{k_m} - u < \epsilon,$$

for all $m > n$. Hence, $(b_{k_n})_{n\in\mathbb{N}}$ converges to u.

Now suppose $(b_{k_n})_{n\in\mathbb{N}}$ is a monotonic decreasing subsequence. The consider instead $(-b_{k_n})_{n\in\mathbb{N}}$ and use the same approach as above.

Q.E.D.