HOMEWORK 7 **DUE: Fri., Nov. 11** **NAME:**

DIRECTIONS:

- Turn in your homework as **SINGLE-SIDED** typed or handwritten pages.
- **STAPLE** your homework together. Do not use paper clips, folds, etc.
- **STAPLE** this page to the front of your homework.
- Be sure to write your name on your homework.
- Show all work, **clearly and in order**.

You will lose point 0.5 points for each instruction not followed.

Questions	Points	Score
1	2	
2	3	
3	1	
4	1	
5	1	
6	1	
7	1	
Total	10	

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Problem 1: (2 points) Find the accumulation points of the following sets in $\mathbb{R}$.

(a) (0.5 points) $S = (0, 1)$.

$$S$$

(b) (0.5 points) $S = \left\{(-1)^n + \frac{1}{n} | n \in \mathbb{N}\right\}$.

$$-1, 1$$

(c) (0.5 points) $S = \mathbb{Q}$.

$$\mathbb{R}$$

(d) (0.5 points) $S = \mathbb{Z}$.

$$\emptyset$$

Problem 2: (3 points)

(a) (1 point) Find an infinite subset of $\mathbb{R}$ that does not have an accumulation point in $\mathbb{R}$.

$$\mathbb{N}$$

(b) (1 point) Find a bounded subset of $\mathbb{R}$ that does not have an accumulation point in $\mathbb{R}$.

$$S = \{1, 2, 3\}$$

(c) (1 point) Find a bounded infinite subset of $\mathbb{Q}$ that does not have an accumulation point in $\mathbb{Q}$.

Use problem 1 on homework 6 to construct a sequence of rational numbers $(s_k)_{k\in\mathbb{N}}$ such that $s_k \to \sqrt{2}$. Then $\sqrt{2}$ is the only accumulation point of the set $S = \{s_k \mid k \in \mathbb{N}\}$ but $\sqrt{2} \notin \mathbb{Q}$.

Problem 3: (1 point) Let $S \subset \mathbb{R}$. Suppose every neighborhood of $x \in S$ contains infinitely many points of S. Prove that x is an accumulation point of S. *(This is the second half of the proof from class.)*

Proof: Suppose every neighborhood of $x \in S$ contains infinitely many points of S. Then $(x - \epsilon, x + \epsilon)$ contains infinitely many points of S. So

$$((x - \epsilon, x + \epsilon) \setminus \{x\}) \cap S \neq \emptyset.$$

whether or not $x \in S$.

Q.E.D.

Problem 4: (1 point) Show that the arbitrary union of open sets in $\mathbb{R}$ is open. That is suppose $\{U_i\}_{i\in\mathcal{I}}$ is a collection of open sets in $\mathbb{R}$. Prove that $\cup_{i\in\mathcal{I}}U_i$ is also open. *Note: $\mathcal{I}$ need not be a denumerable set of indices.*

Proof: Let $\{U_i\}_{i\in\mathcal{I}}$ be a collection of open sets in $\mathbb{R}$. Let $x \in \cup_{i\in\mathcal{I}}U_i$. Then $x \in U_i$ for some $i \in \mathcal{I}$, which is open. Hence, there exists an $\epsilon > 0$ such that

$$(x-\epsilon, x+\epsilon) \subseteq U_i \Longrightarrow (x-\epsilon, x+\epsilon) \subseteq \cup_{i\in\mathcal{I}}U_i.$$

So the arbitrary union is open.

Q.E.D.

Problem 5: (1 point) Show, by example, that an infinite intersection of open sets in $\mathbb{R}$ is not necessarily open (you will still need to prove that your example is not open).

Proof: Consider the sets $U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$. I claim that (i) $\cap_{n\in N}U_n = \{0\}$ and (ii) this set is not open.

(i) Clearly $0 \in U_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$, so $0 \in \cap_{n\in N}U_n$. No let $\epsilon \in \cap_{n\in N}U_n \Longrightarrow \epsilon \in \left(-\frac{1}{n}, \frac{1}{n}\right)$ for all $n \in \mathbb{N}$. But by the ϵ-property, I can always find an $N \in \mathbb{N}$ for all $\epsilon \neq 0$ such that $\epsilon \notin \left(-\frac{1}{n}, \frac{1}{n}\right)$ so ϵ must be zero or we would get a contradiction. Hence $\cap_{n\in N}U_n = \{0\}$.

(ii) $(\cap_{n\in N}U_n)^c = \mathbb{R}\setminus\{0\}$ is open since if we let $x \in \mathbb{R}\setminus\{0\}$ by the ϵ-property there exists an $n \in \mathbb{N}$ such that $|x| > \frac{1}{n} \Longrightarrow x \in \left(x-\frac{1}{n}, x+\frac{1}{n}\right) \subset \mathbb{R}\setminus\{0\}$.

So $(\cap_{n\in N}U_n)^c$ which implies $\cap_{n\in N}U_n$ is closed. However, since $\mathbb{R}$ and $\emptyset$ are the only sets which may be both open and closed, then $\cap_{n\in N}U_n \neq \mathbb{R}, \emptyset$ cannot be open.

Hence, the infinite intersection of open sets need not be open.

Q.E.D.

Problem 6: (1 point) Show, by example, that an infinite union of closed sets in $\mathbb{R}$ is not necessarily closed.

Proof: Consider $U_n = \left[-1, 0-\frac{1}{n}\right]$. Each of these is clearly closed but $A = \cup_{n\in\mathbb{N}}U_n$ is not closed since its complement, A^c will contain 1 (obvious since $1 \notin U_n$ for any n), but no neighborhood of 1 will be contained in A^c by application of the ϵ-property.

Q. E. D.

Problem 7: (1 point) Show that a finite union of closed sets in $\mathbb{R}$ is a closed set in $\mathbb{R}$.

Proof: Let the sets A_n be closed subsets of $\mathbb{R}$. We want to show that $\cup_{n=1}^{k}A_n$ is closed so we will show $\left(\cap_{n=1}^{k}A_n\right)^c$ is open.

$$\left(\cap_{n=1}^{k}A_n\right)^c = \cap_{n=1}^{k}A_n^c,$$

by DeMorgans Law, where A_n^c are each open. Then let $x \in \left(\cap_{n=1}^{k}A_n\right)^c$ so $x \in A_n^c$ for all $n = 1, ..., k$. That means there exists $\epsilon_n > 0$ such that $B_{\epsilon_n} \subseteq A_n$ for each n. Choose $\epsilon = min\{\epsilon_n\}$. Then $B_\epsilon \subseteq A_n$ for each n. Hence $B_\epsilon \subseteq \cap_{n=1}^{k}A_n^c$ so $\left(\cap_{n=1}^{k}A_n\right)^c$ is open.

Q. E. D.