HOMEWORK 8 **DUE: Mon., May 21** **NAME:**

DIRECTIONS:

- Turn in your homework as **SINGLE-SIDED** typed or handwritten pages.
- **DO NOT** staple your homework together. Use a paperclip only.
- Be sure to write your name on **every page** of your homework.
- **Paperclip** this page to the front of your homework.
- Show all work, clearly and in order **You will lose points if any of these instructions are not followed.**

Part I Questions	Points	Score
1	1	
2	1	
3	1	
4	1	
5	2	
6	1	
7	1	
8	1	
9	1	
Total	10	

Problem 1: (1 point) Let F be a field and F^n be a vector space. Prove that the set of canonical basis vectors, $S = \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$, is a linearly independent set.

Proof: Suppose $\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \cdots + \alpha_n\mathbf{e}_n = \mathbf{0}$. Then $(\alpha_1, 0, 0, ..., 0) + (0, \alpha_2, 0, 0..., 0) + \cdots + (0, 0, ..., 0, \alpha_n) = (0, 0, ..., 0)$. Hence by definition of n-tuples in F^n, we must have $\alpha_j = 0$ for all $j = 1, ..., n$.

Q.E.D.

Problem 2: (1 point) Consider the vector space $V = F^n$. Let $\mathbf{v} \in V \setminus \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$. Using only the definition of the canonical vectors, $\mathbf{e}_j$, and the definitions of linearly dependent and independent, prove that $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n, \mathbf{v}\}$ is a linearly dependent set.

Proof: Let $\mathbf{v} \in V \setminus \{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_n\}$. But we may write $\mathbf{v} = (v_1, v_2, ..., v_n)$. So $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + \cdots + v_n\mathbf{e}_n$. Hence

$$1 \cdot \mathbf{v} - v_1\mathbf{e}_1 - v_2\mathbf{e}_2 + \cdots - v_n\mathbf{e}_n = 0,$$

but not all the coefficients are zero (specifically, $1 \neq 0$ is the coefficient of $\mathbf{v}$).

Q.E.D.

Problem 3: (1 point) Consider the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v_m}\}$. Suppose $\mathbf{v}_j = \mathbf{0}$ for some j such that $1 \leq j \leq m$. Prove that this is a linearly dependent set.

Proof: Consider $\alpha_j\mathbf{v}_j + (\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots \alpha_{j-1}\mathbf{v}_{j-1} + \alpha_{j+1}\mathbf{v}_{j+1} + \cdots + \alpha_m\mathbf{v}_m)$.

Let $\alpha_j = 1$ and $\alpha_k = 0$ for all $k \neq j$.

Q.E.D.

Problem 4: (1 point) Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in a vector space V. Show that the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set if and only if one of these vectors is a scalar multiple of the other.

Proof: Suppose that $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a linearly dependent set. Then there exist α_1 and α_2 not both zero such that $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 = \mathbf{0}$. Without loss of generality, suppose $\alpha_1 \neq 0$. Then $\mathbf{v}_1 = \frac{\alpha_2}{\alpha_1}\mathbf{v}_2$.

Now suppose that $\mathbf{v}_1 = c\mathbf{v}_2$. Then $\mathbf{v}_1 - c\mathbf{v}_2 = \mathbf{0}$. Take $\alpha_1 = 1$ and $\alpha_2 = -c$. Then not all the coefficients are zero.

Q.E.D.

Problem 5: (2 points) Determine by inspection if the given set is linearly dependent. Justify your answers.

(a) (0.5 points) $(1, 0)$, $(0, 1)$, $(\sqrt{2}, \pi)$.

Dependent by problem 5.

(b) (0.5 points) $(1, 7, 6)$, $(2, 0, 9)$, $(3, 1, 5)$, $(4, 1, 8)$.

Dependent by problem 5.

(c) (0.5 points) $(2, 3, 5)$, $(0, 0, 0)$, $(1, 1, 8)$.

Dependent by problem 3.

(d) (0.5 points) $(-2,4,6,10)$, $(3,-6,-9,-15)$.

Dependent by problem 4.

Problem 6: (1 point) Let V be a vector space over a field F. Show $\{\mathbf{0}\}$ and V are subspaces of V.

Proof 1: Let $\mathbf{v},\mathbf{w} \in \{\mathbf{0}\} \Longrightarrow \mathbf{v}=\mathbf{w}=\mathbf{0}$. Therefore, $\mathbf{v}+\mathbf{w}=\mathbf{0} \in \{\mathbf{0}\}$. Let $\alpha \in F$. Then $\alpha \cdot \mathbf{v} = \alpha \cdot \mathbf{0} = \mathbf{0} \in \{\mathbf{0}\}$. So $\{\mathbf{0}\}$ is a subspace of V.

Proof 2: Since V is a vector space, it is, by definition, closed under vector addition and scalar multiplication.

Q.E.D.

Problem 7: (1 point) Let $V = \mathbb{Q}[x]$, and let W be the collection of all polynomials in $\mathbb{Q}[x]$ whose degree is less than or equal to a fixed non-negative integer n.

(a) (0.5 points) Prove that W is a subspace of V.

Proof: Let $p,q \in W \subseteq \mathbb{Q}[x]$. Then $p = p_0 + p_1x + p_2x^2 + \cdots + p_nx^n$ and $q = q_0 + q_1x + q_2x^2 + \cdots + q_nx^n$, where $q_i, p_i \in \mathbb{Q}$. Then

$$p + q = (p_0+q_0) + (p_1+q_1)x + (p_2+q_2)x^2 + \cdots + (p_n+q_n)x^n,$$

where $p_i + q_i \in \mathbb{Q}$ since the sum of two rational numbers is a rational number and this is still a polynomial of degree less than or equal to n. Now let $\alpha \in \mathbb{Q}$ then

$$\alpha \cdot p = \alpha p_0 + \alpha p_1 x + \cdots \alpha p_n x^n,$$

where $\alpha p_i \in \mathbb{Q}$ since the product of two rationals is rational and this too is still a polynomial of degree less than or equal to n.

Q.E.D.

(b) (0.5 points) Find the dimension of W and justify your answer.

The dimension of W is $n+1$ since the set $\{1, x, x^2, ..., x^n\}$ forms a basis for W.

Problem 8: (1 point) Let F be a field and consider the vector space $V = F^n$ and for a fixed $m \leq n$, let $W = \{\mathbf{v} \in V |$ v is a linear combination of the basis vectors $\mathbf{e}_1, \mathbf{e}_1, ..., \mathbf{e}_m\}$.

(a) (0.5 points) Prove that W is a subspace of V.

Proof: Let $\mathbf{v},\mathbf{w} \in W$. Then $\mathbf{v} = v_1\mathbf{e}_1 + \cdots v_m\mathbf{e}_m$ and $\mathbf{w} = w_1\mathbf{e}_1 + \cdots w_m\mathbf{e}_m$. So

$$\mathbf{v}+\mathbf{w} = (v_1+w_1)\mathbf{e}_1 + \cdots (v_m+w_m)\mathbf{e}_m \in W.$$

Now let $\alpha \in F$. Then

$$\alpha\mathbf{v} = \alpha v_1 \mathbf{e}_1 + \cdots \alpha v_m \mathbf{e}_m \in W.$$

Q.E.D.

(b) (0.5 points) Find the dimension of W and justify your answer.

The dimension of W is m since $\{\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_m\}$ is a basis for W.

Problem 9: (1 point) Let $V = \mathbb{R}$ be a vector field over $F = \mathbb{R}$. Let $a \in \mathbb{R}$. Consider $T_a : V \to V$ where $T_a(x) = ax$ for all $x \in \mathbb{R}$. Prove that T_a is a linear transformation.

Proof: Let $\mathbf{x}_1, \mathbf{x}_2 \in V = \mathbb{R}$ and $\alpha \in F$. Then

$$T(\mathbf{x}_1 + \mathbf{x}_2) = a(\mathbf{x}_1 + \mathbf{x}_2) = a\mathbf{x}_1 + a\mathbf{x}_2 = T(\mathbf{x}_1) + T(\mathbf{x}_2),$$

since we know that multiplication in $\mathbb{R}$ is distributive over addition. And

$$T(\alpha \cdot \mathbf{x}_1) = a(\alpha \cdot \mathbf{x}_1) = \alpha \cdot (a\mathbf{x}_1) = \alpha \cdot T(\mathbf{x})_1,$$

Since multiplication is commutative in $\mathbb{R}$.

Q.E.D.