HOMEWORK 9 **DUE: Wed., May 30** **NAME:**

DIRECTIONS:

- Turn in your homework as **SINGLE-SIDED** typed or handwritten pages.
- **STAPLE** your homework together. Do not use paper clips, folds, etc.
- **STAPLE** this page to the front of your homework.
- Be sure to write your name on your homework.
- Show all work, **clearly and in order**.

You will lose point 0.5 points for each instruction not followed.

Questions	Points	Score
1	1	
2	1	
3	2	
4	0.5	
5	0.5	
6	1	
7	1	
8	1	
9	1	
10	1	
Extra Credit	1	
Total	10	

Problem 1: (1 point) Let V and W be vector spaces over a field F and $T : V \to W$ a linear transformation.

(a) (0.5 points) Show that $T(\mathbf{0}_V) = \mathbf{0}_W$, where $\mathbf{0}_V$ and $\mathbf{0}_W$ are the additive identities of V and W respectively. *Hint: Use the definition of the additive identity.*

Proof: If $\mathbf{0}_V$ is the additive identity of V, then by definition it must satisfy $\mathbf{0}_V + \mathbf{v} = \mathbf{v} + \mathbf{0}_V = \mathbf{v}$. So $T(\mathbf{0}_V)+T(\mathbf{v}) = T(\mathbf{v})+T(\mathbf{0}_V) = T(\mathbf{v})$ since T is a linear transformation. Hence $T(\mathbf{0}_V)$ satisfies the definition of the additive identity for $\mathbf{w} = T(\mathbf{v}) \in W$.

Q.E.D.

(b) (0.5 points) Show that $T(-\mathbf{v}) = -T(\mathbf{v})$, where $-\mathbf{v}$ is the additive inverse of $\mathbf{v}$ in V and $-T(\mathbf{v})$ is the additive inverse of $T(\mathbf{v})$ in W. *Note, you CANNOT simply say that $T(-\mathbf{v}) = -T(\mathbf{v})$ by the definition of linear transformation because we do not know that '-1' is an element of our field F.*

Proof: If $-\mathbf{v}$ is the additive inverse of $\mathbf{v}$ then we know $-\mathbf{v}+\mathbf{v} = \mathbf{0} = \mathbf{v}-\mathbf{v}$. Hence $T(-\mathbf{v})+T(\mathbf{v}) = T(\mathbf{0}) = T(\mathbf{v}) + T(-\mathbf{v})$ since T is a linear transformation. By part (a), we have

$$T(-\mathbf{v}) + T(\mathbf{v}) = \mathbf{0} = T(\mathbf{v}) + T(-\mathbf{v}),$$

so $T(-\mathbf{v})$ must be the additive inverse of $T(\mathbf{v})$.

Q.E.D.

Problem 2: (1 point) Let V and W be vector spaces over a field F and $T : V \to W$ a linear transformation. Show that $T(V)$ is a subspace of W.

Proof: Let $\mathbf{w}_1, \mathbf{w}_2 \in T(V)$, then there exist $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. Then $\mathbf{w}_1 + \mathbf{w}_2 = T(\mathbf{v}_1) + T(\mathbf{v}_2) = T(\mathbf{v}_1 + \mathbf{v}_2) \in T(V)$ since T is a linear transformation.

Now let $\alpha \in F$. Then $\alpha\mathbf{w}_1 = \alpha T(\mathbf{v}_1) = T(\alpha\mathbf{v}_1) \in T(V)$ since T is a linear transformation.

Q.E.D.

Problem 3: (1.5 points) Let V and W be linearly isomorphic, finite dimensional vector spaces over F. Prove

(a) (0.5 point) $T^{-1} : W \to V$ is a linear transformation.

Proof: Let $\mathbf{w}_1, \mathbf{w}_2 \in W$. Then since T and T^{-1} are bijections, there exist $\mathbf{v}_1, \mathbf{v}_2 \in V$ such that $T^{-1}(\mathbf{w}_i) = \mathbf{v}_i$. Then

$$T^{-1}(\mathbf{w}_1 + \mathbf{w}_2) = T^{-1}\left(T(\mathbf{v}_1) + T(\mathbf{v}_2)\right) = T^{-1}\left(T(\mathbf{v}_1 + \mathbf{v}_2)\right) = \mathbf{v}_1 + \mathbf{v}_2 = T^{-1}(\mathbf{w}_1) + T^{-1}(\mathbf{w}_2).$$

Now let $\alpha \in F$.

$$T^{-1}(\alpha\mathbf{w}_1) = T^{-1}\left(\alpha T(\mathbf{v}_1)\right) = T^{-1}\left(T(\alpha\mathbf{v}_1)\right) = \alpha\mathbf{v}_1 = \alpha T^{-1}(\mathbf{w}_1).$$

Q.E.D.

(b) (0.5 point) $\dim V = \dim W$.

Proof: By part (a) and the theorem from class, we must have $dimV \leq dimW$ and $dimW \leq dimV$, so $dimV = dimW$.

Q.E.D.

(c) (0.5 points) if $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ is a basis for V, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_n)\}$ is a basis for W.

Proof: Let $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ be a basis of V. Then it will be independent as well so

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_n\mathbf{v}_n = \mathbf{0} \iff \alpha_i = 0,$$

for all i. Therefore

$$\alpha_1 T(\mathbf{v}_1) + \cdots + \alpha_n T(\mathbf{v}_n) = T(\mathbf{0}) = \mathbf{0} \iff \alpha_i = 0,$$

Therefore $\{T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_n)\}$ is also linearly independent. Similarly $\{\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_n\}$ spans V, so for all $\mathbf{v} \in V$ there exist $\alpha_i \in F$ such that

$$\mathbf{v} = \alpha_1\mathbf{v}_1 + \cdots \alpha_n\mathbf{v}_n.$$

So since T and T^{-1} are both bijections, for all $\mathbf{w} \in W$, there exist a $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$

$$\mathbf{w} = T(\mathbf{v}) = \alpha_1 T(\mathbf{v}_1) + \cdots \alpha_n T(\mathbf{v}_n).$$

So any $\mathbf{w} \in W$ may be represented as a linear combination of $\{T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_n)\}$. Hence, this is a spanning set as well.

So $\{T(\mathbf{v}_1), T(\mathbf{v}_2), ..., T(\mathbf{v}_n)\}$ is a basis for W.

Q.E.D.

Problem 4: (0.5 points) Let V be a vector space over the field F. If $R, S, T \in \mathcal{L}(V)$, Prove that $R\circ(S+T) = (R \circ S) + (R \circ T)$

Proof: Let $\mathbf{v} \in V$. Then

$$[R \circ (S + T)](\mathbf{v}) = R\,(S(\mathbf{v}) + T(bfv)) = R\,(S(\mathbf{v})) + R\,(T(\mathbf{v})) = (R \circ S)(\mathbf{v}) + (R \circ T)(\mathbf{v}).$$

Q.E.D.

Problem 5: (0.5 points) Let V be a vector space over the field F. Let $I \in \mathcal{L}(V)$ be defined by $I(\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$. Show that $T \circ I = I \circ T = T$ for all $T \in \mathcal{L}(V)$.

Proof: Let $\mathbf{v} \in V$. Then

$$(T \circ I)(\mathbf{v}) = T\,(I(\mathbf{v})) = T(\mathbf{v}) = I\,(T(\mathbf{v})) = (I \circ T)(\mathbf{v}).$$

Q.E.D.

Problem 6: (1 point) Let $\mathbb{R}[x]$ be the vector space of polynomial functions in one variable over $\mathbb{R}$. Define the multiplication of polynomials in the usual way. Show that $\mathbb{R}[x]$ is a commutative algebra with identity.

Proof: We already know that $\mathbb{R}[x]$ is a vector space under the usual scalar multiplication and vector addition. We inherit associativity and distributivity (left and right being equal giving commutativity) of scalar multiplication over addition from the field $\mathbb{R}$. Let 1 be the monomial with $1 \in \mathbb{R}$ of degree zero. Then this is clearly the unit. Hence this is a commutative algebra with unit.

Q.E.D.

Problem 7: (1 point) Show that the number of elements in S_n is $n!$.

Proof: Let $\sigma \in S_n$. We can send 1 to any of n possible elements. However, once we have used one of these elements, we can send 2 to only $n-1$ possible elements. Continue this process and we sent i to one of $n-(i-1)$ possible elements. So the total number of possible permutations is

$$n \cdot (n-1) \cdot (n-2) \cdots (n-(i-1)) \cdots 2 \cdot 1 = n!$$

Q.E.D.

Problem 8: (1 point) Show that the composition of two elements of S_n is also an element of S_n.

Proof: Let $\sigma_1, \sigma_2 \in S_n$. Both are bijections from $\{1, 2, ..., n$ to itself. We proved in chapter 1 that the composition of two bijections is a bijection so $\sigma_1 \circ \sigma_2 \in S_n$.

Q.E.D.

Problem 9: (1 point) Calculate the sign of all elements of S_3 using definition 2.4.4.

Using definition 2.4.4, we find $sgn(I) = sgn(r) = sgn(r^2) = 1$ and $sgn(f_1) = sgn(f_2) = sgn(f_3) = -1$.

Problem 10: (1 point) Show that any $\sigma \in S_n$ can be decomposed into the composition of transpositions.

Proof: Consider $\sigma \in S_n$ such that

$$\sigma = \begin{pmatrix} 1 & 2 & ... & n \\ k_1 & k_2 & ... & k_n \end{pmatrix}.$$

Then since σ is a bijection, there exists an $i_1 \in \{1, 2, ..., n\}$ such that $k_{i_1} = 1$. Compose σ with the transposition $(1\ i_1)$, then

$$\sigma \circ (1\ i_1) = \begin{pmatrix} 1 & 2 & ... & n \\ 1 & k_1 & ... & k_{n-1} \end{pmatrix},$$

by relabeling. Now, since σ is a bijection, there exists an $i_2 \in \{1, 2, ..., n-1\}$ such that $k_{i_2} = 2$. Compose σ with the transposition $(2\ i_2)$, then

$$\sigma \circ (1\ i_1) \circ (2\ i_2) = \begin{pmatrix} 1 & 2 & 3 & ... & n \\ 1 & 2 & k_1 & ... & k_{n-2} \end{pmatrix},$$

by relabeling. Continue this process $n-1$ times to find

$$I = \sigma \circ (1\ i_1) \circ (2\ i_2) \circ \cdots \circ (n\ i_n),$$

where each of these transpositions is either a true transposition or the identity. So

$$\sigma = (i_n\ n) \circ \cdots \circ (1\ i_1),$$

and is the product of transpositions by ignoring the identities.

Q.E.D.

Extra Credit 1: (1 point) Show that $\sigma \in S_n$ is an odd permutation if and only if it is the composition of an odd number of transpositions. *(You may assume without proof that a transposition is an odd permutation.)*

Proof: First suppose that σ is the product of an odd number of transpositions, $h_1, h_2, ..., h_{2n-1}$ for $n \in \mathbb{N}$. Then

$$sgn(\sigma) = sgn(h_1) \cdot sgn(h_2) \cdots sgn(h_{2n-1}) = (-1)^{2n-1} = -1$$

by the proof from class and the assumption above.

From problem 8, we know that every permutation can be written as the composition of a finite number of transpositions, we either have an even number, or an odd number, of such transpositions. Now suppose σ is odd, then $sgn(\sigma) = -1$. Suppose it was the product of an even number of permutations $h_1, h_2, ..., h_{2n}$ for $n \in \mathbb{N}$. Then clearly

$$sgn(\sigma) = sgn(h_1) \cdot sgn(h_2) \cdots sgn(h_{2n}) = (-1)^{2n} = 1$$

by the proof from class and the assumption above which is a contradiction.

Q.E.D.