Math is Delicious, Cooking is a Science

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ABSTRACT:

When opening a business it is crucial to determine the amount required to purchase each product necessary to run the business smoothly while still keeping within a certain budget. It is also important to consider possible price fluctuations in the individual ingredients. The purpose of our project is to develop a budget plan for a business that is about to open, a bakery, which considers the amount of three important products for running the business, flour, sugar, and eggs. The goal of most businesses is to maximize utility and ensure the greatest quality of their product while still minimizing costs. In creating our budget plan we will keep this goal in mind by determining a utility function that models the owner's decisions on the importance of the above mentioned ingredients to the smooth operation of the bakery. The usefulness of amounts $X_{1,}\ldots X_n$ of different goods $G_1\ldots G_n$ can sometimes be measured by a function $U(X_{1\ldots\ldots}X_n)$ called the utility of those goods. In our model we will use these decisions to determine the optimal amounts of each product while still staying within the starting budget as well as leaving room in the model for possible fluctuations that may occur in the process of starting up a new business.

PROBLEM STATEMENT:

We developed a model for purchasing inventory for a bakery that is about to open. The owner of the bakery in question has never run a bakery before, however, and therefore does not know the exact amount of sugar (x), eggs (y), and flour (z) needed. He has instead assigned values to these three ingredients most crucial to the success of his business. He cannot just purchase as much of each as he wants though, as he is constrained by the starting budget he has for inventory. We must take into consideration the amount needed and price fluctuations of these supplies to keep our purchases within a certain budget.

To create this budget plan, we need to create a measure of usefulness of these supplies to the smooth operation of our bakery. To do so we will determine a utility function U that takes into account all three variable supplies. Utility is a measure of the relative satisfaction from, or desirability of, consumption of various goods and services. Thus, a utility function U = f(x1, x2, … xn) means that items x1, x2, etc. to an nth x all contribute to a product's utility, and maximizing this function is a method for determining the overall utility of all the goods.

By maximizing our utility function of our three variables, flour, sugar, and eggs, we will be attaining our goal of maximizing the amount of each good we can purchase within our budget constraint. The accuracy and precision of our budget plan is very important to ensure we efficiently apportion our money and produce our baked goods, and maximize our revenue while minimizing costs. This model aims to find the optimal levels of each ingredient based on his view of their importance and his budgetary restrictions, as well as predict the effects of several shocks to the system as to best prepare him when opening his business. Moreover, the model will be expanded to analyze some production choices once he has purchased his inventory.

MODEL DESIGN

Approach I:

Our first approach to maximize the usefulness of the ingredients involved using specific recipes of two of the bake goods produced in the bakery, represented by equations. The bakery specializes in meringues, r, and cakes, c. Meringue is mainly made of sugar, x, and eggs, y. The recipe of each meringue calls for 3 egg whites and 6 tablespoons of sugar. This recipe can be represented by the equation

$$r = 3y + 6x. \qquad [1]$$

A cake, c, is made mainly of eggs, sugar, and flour, z. The recipe for a cake calls for 1 cup of sugar, 2 egg whites, and 1 ½ cups flour. This recipe can be represented by the equation,

$$c = 2y + x + 1.5z. \qquad [2]$$

Our maximum revenue would then be represented by a utility function where c represents cakes and r represents meringues where utility represents the overall usefulness of the products r and c to the baker. The prices of the cakes and meringues are represented by p_c and p_r respectively. The three ingredients eggs, y, sugar, x, and flour, z, represent the main ingredients in the two recipes and the prices of the ingredients are represented by p_x, p_y, and p_z respectively. The utility function we developed was then:

$$U(x,y) = p_c c + p_r r \qquad [3]$$

with constraint,

$$p_x x + p_y y + p_z z = m. \qquad [4]$$

Using this approach, however, we discovered the function only sought to maximize profit without taking into account the need for all three ingredients and our solutions resulted in a business plan of purchasing a large amount of only one ingredient which, although the best approach financially speaking, did not help our baker who needs all three ingredients to sustain his bakery.

Approach 2:

In our current approach, we still aim to maximize the utility of three specific ingredients that are essential for the running of a bakery. The three ingredients, sugar, eggs, and flour are represented by *x, y,* and *z* respectively. By using a utility function and assigning a level of importance to each ingredient based on the ratio of ingredients used in most recipes, such as one for a yellow cake that calls for ¾ cup of sugar, 1 egg, and 2 cups of flour, the baker is able to develop a way to gather his required information.

His views on the importance of each ingredient are represented by the utility function:

$$U(x,y,z) = \alpha\log(x) + \beta\log(y) + \delta\log(z), \qquad [5]$$

where α, β, and δ are the values assigned to sugar, eggs and flour respectively. The ratio of sugar : eggs : flour is about 1:1:2, so it is possible to assign α, β, and δ specific values of α =1, β=1, and δ=2. The above utility function with the appropriate scalars gives us the following level surfaces :

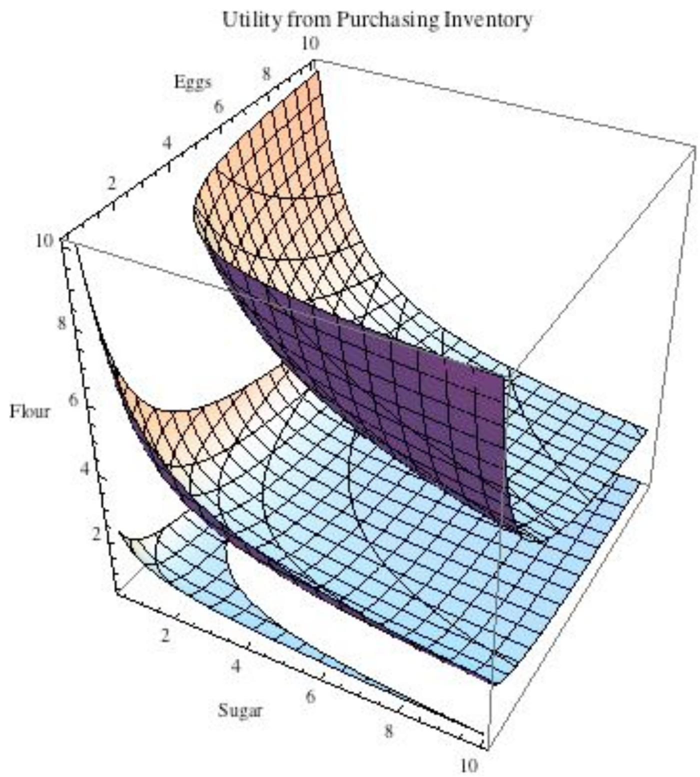

Figure 1: This is the graph showing the increase of the indifference, or utility, level curves of the utility function. The level curves are created using the function U(x,y,z)=log(x) + log(y) + 2log(z). We are interested in finding the highest level of utility and therefore the highest utility curve.

These level curves, known as the indifference curves of the utility function, represent the total utility of each ingredient as they increase in amount. Each curve represents the different combinations of goods that generate the same level of utility, or satisfaction, for the business owner. Therefore, if one variable is changed, for instance suppose x is changed by a factor dx, then y must be changed by dy, and etc. for all other variables in order to maintain the same level of utility. The baker will maximize his utility by choosing the highest indifference curve while remaining within his budget constraint, *m*.

The optimal level of utility, the point at which the budget line and indifference curve are tangent, is what we seek to find. The utility curves we are interested in, as modeled by Figure 1, travel along the gradient of the budget constrain since this is the path of greatest increase to the utility curves. At any different contact point between the utility curves and the budget constraint the relationship between the slopes, and therefore the derivatives, is the relationship between the consumer's own tradeoff and the price ratio, the market's tradeoff. We want to pick a combination where these two conditions are equal, meaning their slopes are equal, which is the tangent. When they are equal the last dollar spent from the budget should yield the same utility no matter on what good it is spent. Any non-tangent points will lie on lower indifference curves, which is undesirable.

Therefore we used the linearization form of the Cobb-Douglas Utility Maximization modeled by equation [5].

This form allows us to show a linear relationship between the inputs and outputs, the goods and utility, up to a scalar coefficient. It transforms a system of equations into a linear relationship by taking log of all variables. This gives us the best approximation due the polynomial structure of the equations with allows us to produce more accurate manipulations with greater ease. This form was first propsed by Knut Wicksell in the 1920's and then tested by Cobb and Douglas in 1928 as they modeled American economic growth from 1899-1922. This form allowed them to have a simplified view of the economy in which production output is determined by the amount of labor and capital invested. We are using similar logic with our model in that utility, or satisfaction output, is determined by the investment of our three ingredients, and therefore this form works the best. After all, we had already attempted using basic polynomials in previous attempts which proved to be unsuccessful.

The utility is constrained by the budget, m, which is dependent on the prices of each good. Let the bakery's budget be *m*. It is the constraint in this problem because we cannot surpass the amount allocated to the budget when purchasing the three ingredients. The constraint equation is then shown by

$$p_x x + p_y y + p_z z = m, \qquad [6]$$

where p_x is the price of sugar per pound, p_y is the price of eggs per dozen, and p_z is the price of flour per pound. The allowed budget, *m*, is set at $1000 for all three ingredients. Using the prices of the ingredients, where the price of sugar= $0.80 per pound, price of eggs= $2.50 per dozen, price of flour= $0.70 per pound and the budget m= $1000, we can see the surface that constrains our utility represented in the budget line graph. This graph takes into account how much of each ingredient the baker can buy with his budget while showing that the more purchased of one ingredient, say sugar, *x*, negatively affects the amount of the other ingredients the baker can buy. The area under the plane represents what is affordable given the budget while the area about the line indicates excess in spending that exceeds the baker's budget. The graph of the budget constraint is given below:

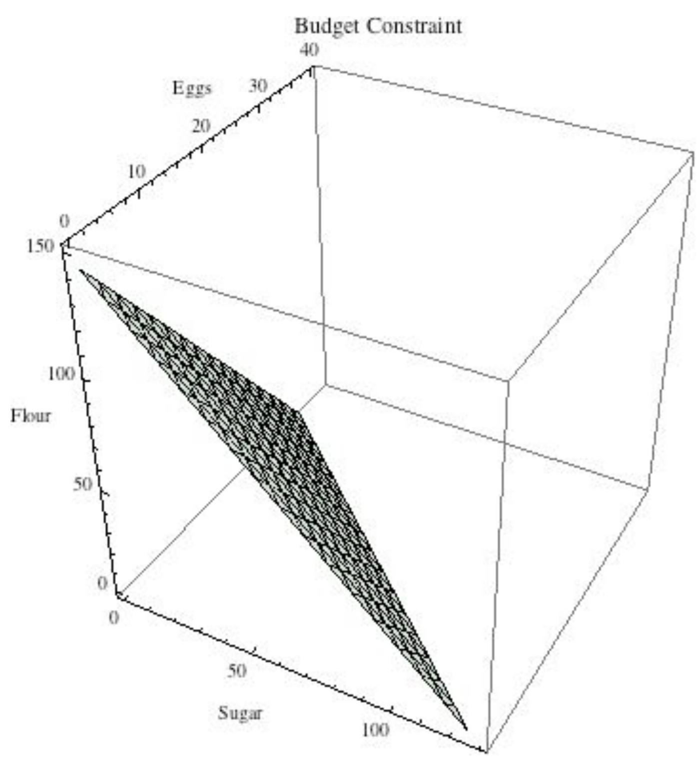

Figure 2: This graph represents the plane that is our budget constraint. This plane stays constant and is the point of reference when determining the highest level of utility. This graph was created using Mathematica and represents the function $m = p_x x + p_y y + p_z z$.

The goal of our model is to find the point where the amount of sugar, eggs, and flour form a utility level surface that is tangent to the budget constraint, as can be seen below using the two surfaces above:

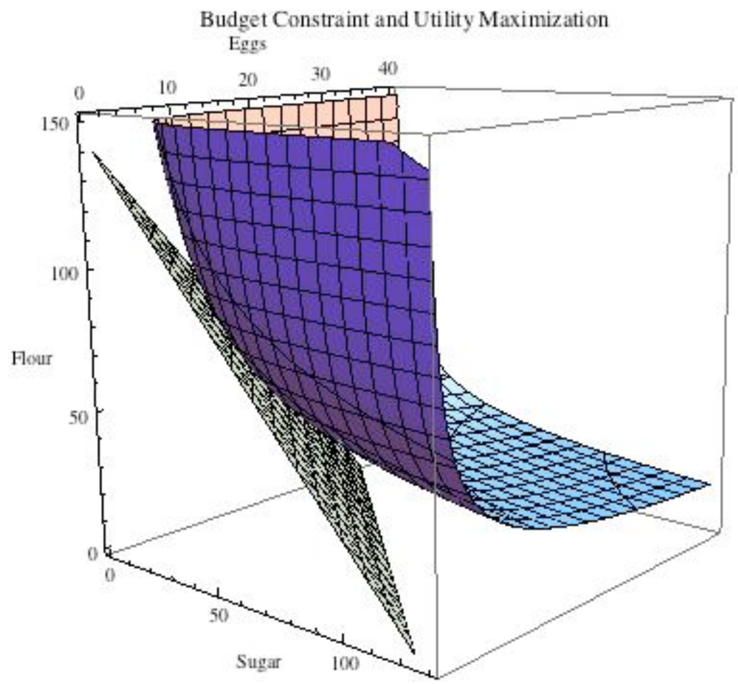

Figure 3: This graph shows the relationship between the highest level of utility and the plane representing the budget constraint. The point at which the two touch is the point of tangency, which is the maximization of our utility function. This graph was made using Mathematic and is a combination of the function represented by equations [5] and [6].

This graph therefore represents the point of maximum utility corresponding to the price of the three specific goods and the budget given to purchase the goods. The baker wants to buy the amount of each good that will give him this tangent relationship. To solve this problem we will use Lagrange Multipliers.

The utility function the baker requires a solution for is therefore represented by

$$\underset{x,y,z}{Max}(U(x,y,z)) = \underset{\alpha,\beta,\delta}{Max}(\alpha\log(x)+\beta\log(y)+\delta\log(z)), \qquad [7]$$

with the constraint represented by equation [5].

It is then possible to derive a Lagrangian function to maximize the amount of each ingredient. This method is appropriate because the baker seeks to maximize the amount of each ingredient he can buy while being subject to a constraint, *m*. By using the utility function, *U(x,y,z)* and the constraint, the Lagrangian function reads,

$$L = U(x,y,z) - \lambda(p_x x + p_y y + p_z z - m) \qquad [8]$$

and

$$\nabla U(x,y,z) = \lambda(p_x x + p_y y + p_z z - m). \qquad [9]$$

By using this equation, we can develop the Lagrange multipliers and solve for the three needed variables, x, y, and z. This method will allow us to find the maximum amount of each ingredient we need to maximize the baker's usefulness of his budget.

After maximization, we must check our values, which can be accomplished by using a bordered Hessian, which is a test that can be used to test critical points with a constraint by using matrices. By inserting our functions into the auxiliary function

$$h = f - \lambda g, \qquad [10]$$

where

$$f = U(x,y,z), \qquad [11]$$

or the utility function equation [5], and

$$g = p_x x + p_y y + p_z z - m, \qquad [12]$$

the constraint, we can take the determinants of the submatrices resulting from the matrix:

$$H(f,g) = \begin{bmatrix} 0 & \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_n} \\ \frac{\partial g}{\partial x_1} & \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1\,\partial x_n} \\ \frac{\partial g}{\partial x_2} & \frac{\partial^2 f}{\partial x_2\,\partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2\,\partial x_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g}{\partial x_n} & \frac{\partial^2 f}{\partial x_n\,\partial x_1} & \frac{\partial^2 f}{\partial x_n\,\partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}, \qquad [13]$$

which is the form of the bordered Hessian.

The conditions of the bordered Hessian are such that if the determinants of the diagonal submatrices are all negative, the point, *h*, is a local minimum and if they start with a positive subdeterminant and then proceed to alternate in sign (>0, <0, >0, etc.), the point is a local maximum.

By taking the partials in this manner and then calculate the determinants of the matrices, we will achieve a value that can be applied to the conditions of the bordered Hessian, and establish definitively if the values are indeed maximum values if the subdeterminants alternate in sign.

MODEL IMPLEMENTATION:

With equation [8], we see that finding the gradients of the components is necessary to use this method. The gradient of the utility function is defined as

$$\nabla U(x,y,z) = \left\langle \frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z} \right\rangle \qquad [14]$$

and the gradient of the budget constraint, BC, is defined

$$\nabla BC = \left\langle \frac{\partial BC}{\partial x}, \frac{\partial BC}{\partial y}, \frac{\partial BC}{\partial z} \right\rangle. \qquad [15]$$

The notation $\frac{\partial U}{\partial x}$ means it is necessary to take the partial derivative of the utility function with respect to x and the notation $\frac{\partial BC}{\partial x}$ represents the partial derivative of the budget constraint equation with respect to x.

Because x is only in the first term of the utility function, the first component of the gradient, ∇U, is equal to the derivative of $\alpha \log x$ which works out to be $\frac{\alpha}{x}$. The same is true for the y and z terms in the function, and the gradient of U(x,y,z) is determined to be:

$$\nabla U(x,y,z) = \left\langle \frac{\alpha}{x}, \frac{\beta}{y}, \frac{\delta}{z} \right\rangle \qquad [16]$$

By using the same method of partials, we can obtain the gradient of the budget constraint function to be:

$$\nabla BC = \langle p_x, p_y, p_z \rangle. \qquad [17]$$

It is then possible to use these gradients to create Lagrange multipliers which make it possible for us to solve for all variables. To set up the multipliers, which can be accomplished by setting the individual terms on each side of the Lagrangian equal to each other, which gives you the three equations:

$$\frac{\partial U}{\partial x} = \lambda\left(\frac{\partial BC}{\partial x}\right), \qquad [18]$$

$$\frac{\partial U}{\partial y} = \lambda\left(\frac{\partial BC}{\partial y}\right), \qquad [19]$$

$$\frac{\partial U}{\partial z} = \lambda\left(\frac{\partial BC}{\partial z}\right). \qquad [20]$$

By plugging in the components of the respective gradients, we get:

$$\frac{\alpha}{x} = \lambda(p_x), \qquad [21]$$

$$\frac{\beta}{y} = \lambda(p_y), \qquad [22]$$

$$\frac{\delta}{x} = \lambda(p_z), \qquad [23]$$

with constraint,

$$p_x x + p_y y + p_z z = m. \qquad [24]$$

To solve for *x, y,* and *z*, it is possible to first divide equation [21] by [22] then [21] by [23] and rearrange to get the variable in question in terms of *x* as follows:

$$\frac{y}{x}\frac{\alpha}{\beta} = \frac{p_x}{p_y} \Rightarrow y = \frac{p_x}{p_y}\frac{\beta}{\alpha}x \quad , \qquad [25]$$

$$\frac{z}{x}\frac{\alpha}{\delta} = \frac{p_x}{p_z} \Rightarrow z = \frac{p_x}{p_z}\frac{\delta}{\alpha}x \quad . \qquad [26]$$

By using these solved *y* and *z*-valued equations, it is possible to then insert equations [20] and [21] into equation [5] to solve for *x* as follows:

$$p_x x + p_y\left(\frac{p_x}{p_y}\frac{\beta}{\alpha}x\right) + p_z\left(\frac{p_x}{p_z}\frac{\delta}{\alpha}x\right) = m \Rightarrow p_x x + \frac{\beta}{\alpha}p_x x + \frac{\delta}{\alpha}p_x x = m \Rightarrow x\left(1 + \frac{\beta}{\alpha} + \frac{\delta}{\alpha}\right) = \frac{m}{p_x} \quad , \quad [27]$$

$$\Rightarrow x^* = \frac{m}{p_x}\left(\frac{\alpha}{\alpha + \beta + \delta}\right), \qquad [28]$$

$$\Rightarrow y^* = \frac{m}{p_y}\left(\frac{\beta}{\alpha + \beta + \delta}\right), \qquad [29]$$

$$\Rightarrow z^* = \frac{m}{p_z}\left(\frac{\delta}{\alpha + \beta + \delta}\right), \qquad [30]$$

where x*, y*, and z* represent the optimal amount of sugar, eggs, and flour purchased by the baker respectively.

To find the actual values of the maximum amounts of each ingredient we can apply the prices of sugar, eggs and flour which are represented by P_x,, P_y, and P_z respectively. The values that were previously established are P_x=\$0.80, P_y= \$2.50, and P_z=\$0.70. We can also use \$1000 for m because m represents the budget constraint, which is set at \$1000. Finally, we plug in α=1, β=1 and δ=2 which we determined as the coefficients of our utility function based on the ratios of our three ingredients needed in a recipe for cake.
Therefore,

$$\Rightarrow x^* = \frac{1000}{.8}\left(\frac{1}{1+1+2}\right), \qquad [31]$$

$$\Rightarrow y^* = \frac{1000}{2.5}\left(\frac{1}{1+1+2}\right), \qquad [32]$$

$$\Rightarrow z^* = \frac{1000}{.7}\left(\frac{2}{1+1+2}\right). \qquad [33]$$

With basic addition and multiplication, we obtain our values, which are:

x= 312.5 lbs. of sugar,*

y= 100 dozen eggs,*

z= 357.1 lbs. of flour.*

We now wish to verify that these values indeed satisfy the desired maximization. For this we can use a bordered Hessian to determine that the Cobb-Douglas utility function does indeed always produce a maximum value. For this bordered Hessian, we use equation [10] to produce:

$$h = \alpha \log(x) + \beta \log(y) + \delta \log(z) - \lambda(p_x x + p_y y + p_z z - m). \qquad [34]$$

Because we know equation [5] and equation [6] are both positive, and all values $x, y, z, Px, Py, Pz, \alpha, \beta, \delta$ are >0, we are able to then use equation [13] to produce the following matrix:

$$\begin{bmatrix} 0 & -p_x & -p_y & -p_z \\ -p_x & \frac{-\alpha}{x^2} & 0 & 0 \\ -p_y & 0 & \frac{-\beta}{y^2} & 0 \\ -p_z & 0 & 0 & \frac{-\delta}{z^2} \end{bmatrix}, \qquad \text{[Matrix 1]}$$

This can then create the submatrix:

$$\begin{bmatrix} 0 & -p_x & -p_y \\ -p_x & \frac{-\alpha}{x^2} & 0 \\ -p_y & 0 & \frac{-\beta}{y^2} \end{bmatrix} \quad , \qquad \text{[Matrix 2]}$$

what follows by taking the determinant is then:

$$0\left(\frac{\alpha\beta}{x^2y^2}\right) - (-)\ p_x\left(\frac{p_x\beta}{y^2}\right) + (-)p_y\left(\frac{\alpha p_y}{x^2}\right),$$

which produces a value that is always >0 because all the variables >0.

By solving for the rest of the bordered Hessian, we discover:

$$p_x \det\begin{bmatrix} -p_x & 0 & 0 \\ -p_y & \frac{-\beta}{y^2} & 0 \\ -p_z & 0 & \frac{-\delta}{z^2} \end{bmatrix} + \frac{-\alpha}{x^2}\det\begin{bmatrix} 0 & -p_y & -p_z \\ -p_y & \frac{-\beta}{y^2} & 0 \\ -p_z & 0 & \frac{-\delta}{z^2} \end{bmatrix} = -p_x^2\left(\frac{\beta\delta}{y^2z^2}\right) - \frac{\alpha}{x^2}\left[p_y\left(\frac{p_y\delta}{z^2}\right) + -p_z\left(\frac{-p_z}{y^2}\right)\right]$$

$$= -p_x^2\left(\frac{\beta\delta}{y^2z^2}\right) - \frac{\alpha}{x^2}\left[\frac{p_y^2\delta}{z^2} + \frac{p_z^2\beta}{y^2}\right]$$

Which is <0 because all variables are again positive.

Therefore, the signs of the matrices alternate, and the utility function always produces a maximum value, meaning the values we produced are indeed maximum utility values for the specified ingredients. This validates the claim that the amounts x*, y*, and z* will indeed maximize the utility of the sugar, eggs, and flour purchased by the baker.

DISCUSSION:

One possible weakness of our model is that our model only works for a linear constraint and would not necessarily work for a nonlinear constraint. Also, we only considered a maximization for the starting up of the bakery, which does not necessarily take into account weekly or monthly demands for each of these ingredients. A further application of our model could be to develop a model in which the constraint varies, for instance based on a weekly profit of the bakery. Also, we could add to our already existing model by considering the maximization of revenue based on equations that represent recipes for specific goods that the bakery produces. This would be very helpful for the bakery in further stages past the beginning stage, which our model focuses on.

However, our model does represent an efficient budget plan for the bakery. It accounts for possible market price fluctuations of each of the three ingredients. To account for this change in price all we would have to do is change the values of p_x, p_y, and p_z in our final equation, which represent the prices of the ingredients. Second, because of the equation that we chose to represent the utility function there is an ease of communication possible through simple visualizations (Figures 1, 2 and 3). This gives our model more clarity and merit since audiences without a diverse mathematical background have an opportunity to use our model. It may also be generalized to include more than three variables since we would just have to add more terms to our log function representing the linear form of the Cobb-Douglas Utility function. Third, our model can be adapted by other businesses, not just a bakery. This model could apply to any business trying to determine how much of a specific product is needed in respect to a budget constraint. Finally, our model also allows us to confirm using a Hessian that the result we attained is indeed a maximum.

CONCLUSION:

In our model we successfully developed a model for purchasing inventory for a newly opening bakery. To develop the budget plan we created a measure of usefulness of three crucial supplies for the bakery, sugar, flour, and eggs, by determining a utility function. This utility function took into account all three of these variables and represented the relative satisfaction from the consumption of these goods. By maximizing this function we were able to maximize the amount of these goods that could be purchased within a budget constraint, in our case $1000. To determine the maximum amounts of these goods we placed an equal importance on all three ingredients in consideration of this constraint and were able to come up with reasonable amounts of all three goods, 312.5 pounds of sugar, 100 dozen eggs, and 357 pounds of flour. This result seems like a decent starting amount of each ingredient for the business. Our model not only came up with a reasonable amount for the bakery, but can be generalized to apply to other business, can be expanded to include more variable, and can account for market price fluctuation among all of the variables in question.

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