Homework for Week 11 **Math 231 Fall 2001**

This homework will not be collected. It is your responsibility to do as many problems as necessary to understand the material (this includes doing extra problems if you need more practice). We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 4.4 3, 9, 11, 14, 16, 18, 19, 30, 32, 36, 37, 39, 43, 45, 50, 53, 58, 59, 60.

Section 4.5 2, 7, 10, 11, 13, 19, 25, 30, 34, 36, 39, 45, 52, 53, 56, 58, 59, 65.

Section 5.1 8, 10, 11, 21, 23, 28, 33, 42, 43, 47, 52, 53, 59, 65, 71, 75, 78, 81, 84, 90, 95.

Selected Hints and Answers

Caution: The hints and answers below are not necessarily full solutions. Many of them would not be considered complete on a quiz or test.

Section 4.4

3. Domain is $(-\infty, 0) \cup (0, \infty)$, range is $(-\infty, 0)$. f is positive on $(-\infty, 0) \cup (0, \infty)$ and does not exist at $x = 0$. f is decreasing on $(-\infty, 0)$ and increasing on $(0, \infty)$. f is concave down on $(-\infty, 0) \cup (0, \infty)$. f has a vertical asymptote at $x = 0$ and a horizontal asymptote at $y = 0$. f has no roots, no local or global extrema, and no inflection points.

11. $(0, \infty)$ if k is even, $(-\infty, 0) \cup (0, \infty)$ if k is odd.

16. Part (a): If k is a positive integer and $A > 0$ then $y = Ax^k = (\sqrt[k]{A})^k x^k = (\sqrt[k]{A}x)^k$, which is a horizontal stretch of $y = x^k$ by a factor of $\sqrt[k]{A}$.

32. $f(-x) = -2(-x)^{-1} = \frac{-2}{-x} = \frac{2}{x} = 2x^{-1} = -f(x)$, so f is an odd function.

36. Two possible functions are $y = \frac{-2}{x}$ and $y = \frac{-2}{x^3}$.

37. Two possible functions are $y = 2x^2$ and $y = \frac{1}{2}x^4$.

39. $y = 2x^4$.

43. $\frac{1}{5}x^2$.

45. The function $g(x) = \frac{1}{f(x)}$ has values $g(-2) = -\frac{1}{4}$, $g(1) = -2$, $g(0)$ DNE, $g(1) = 2$, and $g(2) = \frac{1}{4}$. The graph of g has a horizontal asymptote at $y = 0$ and a vertical asymptote at $x = 0$.

50. Domain is $(-\infty, \infty)$, range is $(-\infty, \infty)$. $f(x) = 0$ at $x = 0$, $f(x) < 0$ on $(-\infty, 0)$, $f(x) > 0$ on $(0, \infty)$. $f'(x) = 0$ at $x = 0$, $f'(x) > 0$ on $(-\infty, 0) \cup (0, \infty)$. $f''(x) = 0$ at $x = 0$, $f''(x) < 0$ on $(-\infty, 0)$, $f''(x) > 0$ on $(0, \infty)$. $\lim_{x\to\infty} f(x) = \infty$, $\lim_{x\to-\infty} = -\infty$.

58. **Proof:** If k is an odd integer and $f(x) = x^k$, then $f(-x) = (-x)^k = (-1 \cdot x)^k = (-1)^k x^k = (-1)x^k = -x^k = -f(x)$, so f is an odd function. (The third equality is an algebraic property of exponents; the fourth equality uses the fact that k is odd, since in that case $(-1)^k = -1$.) ■

59. **Proof:** $x^6 \le x^4 \Leftrightarrow x^6 - x^4 \le 0 \Leftrightarrow x^4(x^2 - 1) \le 0 \Leftrightarrow x^2 - 1 \le 0 \Leftrightarrow |x| < 1$. (The second to last "if and only if" follows from the fact that x^4 is always positive.) The proof of the second part is similar. ■

60. Follow the same method as the proof on pp.433-434.

Section 4.5

11. TYPO: The problem should say that $\frac{p}{q}$ is positive; parts (a) and (b) are slightly different if the power is negative.
Part (a): If q is even then $x^{\frac{p}{q}}$ is not defined for negative values of x, since you can't take an even root of a negative number. However you *can* take an even root of a positive number (or zero), and of course you can raise any number to the p^{th} power. Therefore the domain of $x^{\frac{p}{q}}$ is $[0, \infty)$.

30. $f(-x) = -0.25(-x)^{-\frac{5}{7}} = -0.25(-1)x^{-\frac{5}{7}} = 0.25x^{-\frac{5}{7}} = -f(x)$, so f is an odd function.

34. Domain is $(-\infty, \infty)$, range is $[0, \infty)$. $f(x) = 0$ at $x = 0$, $f(x) > 0$ on $(-\infty, 0) \cup (0, \infty)$. $f'(x)$ does not exist at $x = 0$, $f'(x) < 0$ on $(-\infty, 0)$, $f'(x) > 0$ on $(0, \infty)$. $f''(x)$ does not exist at $x = 0$, $f''(x) < 0$ on $(-\infty, 0) \cup (0, \infty)$. $\lim\limits_{x\to-\infty} f(x) = \infty$, $\lim\limits_{x\to\infty} f(x) = \infty$ (so no horizontal asymptotes). $\lim_{x\to 0^-} f'(x) = -\infty$, $\lim_{x\to 0^+} f'(x) = \infty$ (so f has a vertical cusp at $x = 0$).

39. $f(x) = -3x^{\frac{2}{5}}$ has domain $(-\infty, \infty)$ and range $[0, \infty)$. $\frac{1}{f(x)} = -\frac{1}{3}x^{-\frac{2}{5}}$ has domain $(-\infty, 0) \cup (0, \infty)$ and range $(0, \infty)$.

45. $f(x)$ is not one-to-one, but its restriction to the domain $[0, \infty)$ is one-to-one. On this restricted domain its inverse is $f^{-1}(x) = -\frac{1}{3}x^{\frac{5}{2}}$.

52. Two possible functions are $y = -\frac{1}{4}x^{\frac{2}{3}}$ and $y = (-8^{-\frac{4}{5}})x^{\frac{4}{5}}$.

53. $y = 2x^{\frac{1}{4}}$.

56. Part (a): The graph looks a little funny at $x = 0$; maybe it's "pointy" there? But if you zoom in near $x = 0$ you will see that in fact it looks like there is a horizontal tangent line at $x = 0$.
Part (b): If $f(x) = x^{\frac{6}{5}}$ then $f'(0) = \lim\limits_{h\to 0} \dfrac{(0+h)^{\frac{6}{5}} - 0^{\frac{6}{5}}}{h} = \lim\limits_{h\to 0} \dfrac{h^{\frac{6}{5}}}{h} = \lim\limits_{h\to 0} h^{\frac{1}{5}} = 0$. (Also notice that $f'(x) = \frac{6}{5}x^{\frac{1}{5}}$ exists, and is zero, at $x = 0$.)

59. The graph of $g(x) = \frac{1}{f(x)}$ has values $g(-8) = -\frac{1}{6}$, $g(-1) = -\frac{1}{3}$, $g(1) = \frac{1}{3}$, and $g(8) = \frac{1}{6}$. This reciprocal function g is not defined at $x = 0$, and in fact has a vertical asymptote there (as well as a horizontal asymptote at $x = 0$).
The graph of $f^{-1}(x)$ has values $f^{-1}(-6) = -8$, $f^{-1}(-3) = -1$, $f^{-1}(0) = 0$, $f^{-1}(3) = 1$, and $f^{-1}(6) = 8$. Its graph has a horizontal tangent line at $x = 0$ and can be obtained by reflecting the graph of $f(x)$ over the line $y = x$.

Section 5.1

8. $f(x) = x(x+1)(x-2)^2 = x^4 - 3x^3 + 4x$ is a polynomial function (a quartic), with leading coefficient 1, leading term x^4, degree 4, and constant term 0, with $a_1 = 4$ and $a_3 = -3$.

11. Part (a): $f(x) = x(2x^2 - 3x + 8)$. The quadratic term is irreducible because $b^2 - 4ac = (-3)^2 - 4(2)(8) < 0$.
Part (b): $f(x) = 2x(x^2 - \frac{3}{2}x + 4)$.

23. $f(-1) = 0$, $f(1) = 0$, and $f(\frac{1}{2}) = 0$, but $f(2) = 9 \neq 0$ and $f(-3) = 224 \neq 0$. Notice that you did *not* have to factor the polynomial to answer this question!

28. One example is $f(x) = x(x-2)(x+5)$. Another is $f(x) = 5x(x-2)(x+5)$. Another is $f(x) = x(x-2)^2(x+5)^3$.

33. One example is $f(x) = (x-1)^2(x+3) = x^3 + x^2 - 5x + 3$.

47. (a) One example is $f(x) + 2$; (b) $f(x) + 1.5$; (c) $f(x)$; (d) $f(x) - .4$ or $f(x) - 1.8$; (e) one example is $f(x) - 1$.

52. True.

53. False.

Section 5.1 (continued)

59. $f(x) = 3(x+1)(x-2)+8 = 3x^2 - 3x + 2$, which has possible integer roots ± 1 and ± 2. However, *none* of these possible roots are actually roots of f (since none of them make $f(x) = 0$; in fact, $f(x)$ is an irreducible quadratic).

65. The only possible integer roots are ± 1, ± 3, and ± 9. Of these possible roots, only $x = -1$ and $x = 3$ are actually roots. Since we are told that the polynomial has only integer roots, these are the only two roots of $f(x)$; thus one of the roots must be repeated. Try multiplying out $(x+1)^2(x-3)$ and $(x+1)(x-3)^2$ to see which one works.

71. $f(x) = (x^2 - 16)(1 + 2x) = (x+4)(x-4)(1+2x)$.

75. $f(x) = 2x(x^2 - x - \frac{1}{2})$.

78. $f(x) = 3(x - \frac{2}{3})(x^2 + 4)$.

81. Possible integer roots are ± 1, ± 2, ± 3, and ± 6; of these possible roots only $x = 1$ and $x = -6$ are actually roots. After using synthetic division with $c = 1$ (and then factoring a quadratic), we get $f(x) = (x-1)^2(x+6)$.

84. $f(x) = 2(x-1)(x+1)(4+x^2)$.

90. Using synthetic division with $c = -1$ we get the polynomial $5x^3 - 5x^2 + 5x - 8$, with a remainder of 10. Thus $\dfrac{f(x)}{x+1} = (5x^3 - 5x^2 + 5x - 8) + \dfrac{10}{x+1}$. In other words, $f(x) = (5x^3 - 5x^2 + 5x - 8)(x+1) + 10$.

95. The possible rational roots are $\frac{p}{q}$ where p is ± 1 and $q = \pm 1, \pm 2, \pm 4, \pm 8$, or ± 16; in other words the possible rational roots are $\frac{p}{q} = \pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}, \pm\frac{1}{8}$, and $\pm\frac{1}{16}$. Checking these possible roots we see that only $\frac{p}{q} = \frac{1}{2}$ and $\frac{p}{q} = -\frac{1}{4}$ are roots of $f(x) = 16x^3 - 12x^2 + 1$. This means that $2x - 1$ and $4x + 1$ are factors of $f(x)$. Using synthetic division with $c = \frac{1}{2}$ we find that $f(x) = (2x-1)(8x^2 - 2x - 1) = (2x-1)(2x-1)(4x+1) = (2x-1)^2(4x+1)$.