Homework for Week 13 **Math 231 Fall 2001**

This homework will not be collected. It is your responsibility to do as many problems as necessary to understand the material (this includes doing extra problems if you need more practice). We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 6.1 2, 4, 8, 14, 15, 17, 20, 21, 23, 28, 29, 31, 34, 35, 42, 43, 53, 55.

Section 6.2 4, 6, 12, 13, 14, 15, 19, 24, 25, 26, 36, 37, 38, 42, 43, 46, 52, 56, 58, 59.

Selected Hints and Answers

Caution: The hints and answers below are not necessarily full solutions. Many of them would not be considered complete on a quiz or test.

Section 6.1

4. $f(x) = \dfrac{-x^3 + 2x + 3}{x^2 + 3x}$

8. $f(x) = \dfrac{3x^2 - x + 2}{x^3}$

14. The graph should be just like $y = x + 1$, but with holes at $x = -2$ and $x = 3$.

15. f has roots at $x = 1$ and $x = -3$, and the domain of f is $(-\infty, 2) \cup (2, 3) \cup (3, 4) \cup (4, \infty)$.

21. $f(x) = \dfrac{(x-1)^2(x-3)}{(x-3)(x+2)}$ and $f(x) = \dfrac{(x-1)(x-3)(x^2+1)}{(x-3)(x+2)}$ are two possible answers.

23. One possible answer is $f(x) = \dfrac{(x-3)^2}{(x-3)}$.

28. $p(x) = q(x)(x^2 - x + 3) + (3x + 1)$, $\dfrac{p(x)}{q(x)} = (x^2 - x + 3) + \dfrac{3x - 1}{q(x)}$.

31. False.

34. True.

35. False.

42. $f(x) = \dfrac{(x-2)^3}{-(x-3)(x-2)(x+2)}$ (synthetic division for the numerator, grouping for the denominator), so the domain of f is $(-\infty, -2) \cup (-2, 2) \cup (2, 3) \cup (3, \infty)$, f has a hole at $x = 2$, and f has no roots.

43. $f(x) = \dfrac{(x-1)(x+2)(x+5)}{(x-1)(x+1)(x+3)(x^2+1)}$ (synthetic division in numerator and denominator), so the domain of f is $(-\infty, -3) \cup (-3, -1) \cup (-1, 1) \cup (1, \infty)$, f has a hole at $x = 1$, and f has roots at $x = 2$ and $x = 5$.

53. $f(x) = (\frac{1}{2}x^2 - \frac{7}{4}) + \dfrac{\frac{15}{4}}{2x^2 + 1}$.

55. $f(x) = (x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6) + \dfrac{7x - 7}{x^2 - 2x + 1}$.

Section 6.2

4. Two possible answers are $f(x) = \dfrac{x}{(x+1)(x-1)}$ and $f(x) = \dfrac{(x+1)(x-1)}{x^3}$.

6. (a) $n \leq m$; (b) $n < m$; (c) $n = m + 1$; (d) $n \geq m + 2$; (e) $n = m + 4$.

12. One possible answer is $f(x) = \dfrac{3x^2 + 1}{(x+2)(x-2)}$.

Section 6.2 (continued)

13. TYPO: Should say $y = 5$, not $x = 5$.
One possible answer is $f(x) = \dfrac{-5(x+2)^3}{(x+2)(x-1)(x+3)}$.

14. One possible answer is $f(x) = \dfrac{x(x^2+1)}{x(x-2)}$.

15. One possible answer is $f(x) = (x^3+2) + \dfrac{1}{x-1} = \dfrac{x^4 - x^3 + 2x - 1}{x-1}$.

19. False.

24. $\frac{3}{4}$.

25. 0.

26. The limit is infinite; specifically, $-\infty$ as $x \to 2^-$ and ∞ as $x \to 2^+$.

36. 1.

37. ∞.

38. 0.

42. f has a root at $x = 2$, vertical asymptotes at $x = 1$ and $x = -\frac{2}{3}$, and a horizontal asymptote at $y = 0$.

43. $f(x) = \dfrac{(2x-1)(3x+5)}{x(x-1)(x+2)}$ (synthetic division for the denominator), so f has roots at $x = \frac{1}{2}$ and $x = -\frac{5}{3}$, vertical asymptotes at $x = 0$, $x = 1$, and $x = -2$, and a horizontal asymptote at $y = 0$.

46. $f(x) = \dfrac{(x-2)(x+2)^2}{(x-1)(2x^2+x+1)}$ (grouping for the numerator, synthetic division for the denominator), so f has roots at $x = \pm 2$, a vertical asymptote at $x = 1$, and a horizontal asymptote at $y = \frac{1}{2}$.

52. f has roots at $x = \pm 1$ and $x = -\frac{2}{3}$, no holes, and no vertical asymptotes (the quadratic in the denominator is irreducible). By polynomial long division we have $f(x) = (3x-1) + \dfrac{-5x-1}{x^2+x+1}$, so f has a slant asymptote with equation $3x - 1$.

56. $f(x) = \dfrac{x^3(x-1)^2}{(x-1)(x+1)}$, so f has a root at $x = 0$, a hole at $x = 1$, and a vertical asymptote at $x = -1$. By polynomial long division we can also write $f(x) = (x^3 - 2x^2 + 2x - 2) + \dfrac{2x-2}{x^2-1}$, so f has a cubic curve asymptote with equation $y = x^3 - 2x^2 + 2x - 2$.

58. Suppose $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}$ is a rational function. In each of the three cases we have:

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0}\left(\frac{1/x^m}{1/x^m}\right) \\ &= \lim_{x\to\infty} \frac{a_n x^{n-m} + a_{n-1}x^{n-1-m} + \ldots + a_1 x^{1-m} + a_0 x^{-m}}{b_m + b_{m-1}x^{-1} + \ldots + b_1 x^{1-m} + b_0 x^{-m}}.\end{aligned}$$

Part (a): If $n < m$, then each of the terms in the numerator has a negative exponent (since $n < m$) and thus goes to zero as $x \to \infty$. Each term after the first one in the denominator also goes to zero as $x \to \infty$. Thus the limit above $\to \frac{0}{b_m} = 0$ as $x \to \infty$. A similar argument shows that $\lim\limits_{x\to-\infty} f(x) = 0$. Therefore f has a horizontal asymptote at $y = 0$.
Parts (b) and (c) are similar, using the same limit calculation.

59. Part (a): We have $\dfrac{p(x)}{q(x)} = \dfrac{l(x)q(x) + R(x)}{q(x)}$, so $l(x)q(x)$ must have the same degree as $p(x)$ (say degree n). Since we know that the degree of $q(x)$ is $n-1$, therefore the degree of $l(x)$ must be 1 (*i.e.* $l(x)$ is a linear function).

Section 6.2 (continued)

59. Part (b): The polynomial long division algorithm terminates when the degree of the remainder is less than the degree of $q(x)$; therefore the degree of $R(x)$ is less than the degree of $q(x)$, so $\frac{R(x)}{q(x)}$ is a proper rational function.

Part (c): $\lim_{x \to \infty} \left(\frac{p(x)}{q(x)} - l(x) \right) = \lim_{x \to \infty} \frac{R(x)}{q(x)} = 0$ because $\frac{R(x)}{q(x)}$ is a proper rational function. The calculation for the limit as $x \to -\infty$ is similar.

Part (d): We have just shown that as $x \to \infty$ (and as $x \to -\infty$), the difference between the rational function $f(x) = \frac{p(x)}{q(x)}$ and the linear function $l(x)$ goes to zero. This means that $f(x) = \frac{p(x)}{q(x)}$ gets closer and closer to the line $l(x)$ as $x \to \infty$ (and as $x \to -\infty$), *i.e.* that $l(x)$ is a slant asymptote of $f(x) = \frac{p(x)}{q(x)}$.