Homework for Week 14 **Math 231 Fall 2001**

This homework will not be collected. It is your responsibility to do as many problems as necessary to understand the material (this includes doing extra problems if you need more practice). We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 6.3 1, 2, 3, 4, 5, 6, 10, 14, 20, 29, 31, 33, 40, 45, 48, 49, 52.

Section 6.4 1, 3, 4, 5, 6, 7, 8, 9, 16, 17, 23, 24, 31, 32, 33, 35, 37.

Selected Hints and Answers

Caution: The hints and answers below are not necessarily full solutions. Many of them would not be considered complete on a quiz or test.

Section 6.3

2. $f'(x) = \dfrac{(\frac{1}{2}x^{-\frac{1}{2}} - 6x^{-1})(x^{\frac{3}{2}}+1) - (x^{\frac{1}{2}} + 3x^{-2})(\frac{3}{2}x^{\frac{1}{2}})}{(x^{\frac{3}{2}}+1)^2}$

3. Part (a): The numerator has degree $n+m-1$ and the denominator has degree $2m$.
   Part (b): Yes, since if $n < m$ then $n+m-1 < m+m-1 = 2m-1 < 2m$.
   Part (c): Yes, at $y = 0$.
   Part (d): The degree of the numerator must be exactly two more than the degree of the denominator (then the numerator of the derivative will have a degree exactly one more than the degree of the numerator). Show this using part (a) and $n = m+2$.

4. $f(x) = \dfrac{x^3 - x + 5}{x^2 + 3x - 1}$.

5. Part (a): Global minimum at $x = -1$, global maximum at $x \approx \frac{1}{2}$.
   Part (b): No global minimum, global maximum at $x \approx \frac{1}{2}$.
   Part (c): No global minimum, global maximum at $x \approx \frac{1}{2}$.
   Part (d): No global maximum or minimum.
   Part (e): Global minimum at $x = 5$, no global maximum.

6. $f'(2) = \lim_{h\to 0} \dfrac{f(2+h) - f(2)}{h} = \lim_{h\to 0} \dfrac{\frac{(2+h)-1}{(2+h)+3} - \frac{2-1}{2+3}}{h} = \ldots(\text{work})\ldots = \frac{4}{25}$.

10. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\frac{(x+h)-1}{(x+h)-3} - \frac{x-1}{x+3}}{h} = \ldots(\text{work})\ldots = \dfrac{4}{(x+3)^2}$.

14. $f'(x) = \lim_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \dfrac{\frac{(x+h)^3}{(x+h)+1} - \frac{x^3}{x+1}}{h} = \ldots(\text{work})\ldots = \dfrac{2x^3 + 3x^2}{(x+1)^2}$.

20. $f'(x) = \dfrac{(7x^6 - 15x^4)(1-3x^4) - (x^7 - 3x^5 + 4)(-12x^3)}{(1-3x^4)^2}$.

29. After multiplying out $f(x)$, differentiating, simplifying the derivative, and factoring (this requires synthetic division), you get $f'(x) = \dfrac{x(x-2)(x^2-6x+4)}{(x-2)^4}$. The quadratic in the numerator is *not* irreducible, but its roots are messy; by the quadratic formula they are $x = 3 \pm \sqrt{5}$. Thus $f'(x)$ is zero at $x = 0$ and $x = \pm\sqrt{5}$, and is not defined at $x = 2$.

31. After simplifying and factoring, $f'(x) = \dfrac{-3(2x+1)}{(x-1)^2(x+2)^2}$. Thus the critical points of f are $x = -2$, $x = -\frac{1}{2}$, and $x = 1$. The number line for $f'(x)$ has signs $+$, $+$, $-$, $-$ (in that order); thus $x = -\frac{1}{2}$ is the only local extrema (a maximum).

Section 6.3 (continued)

33. After simplifying and factoring, $f'(x) = \dfrac{-2(x-2)}{(x-2)^4}$. Thus the only critical point of f is $x = 2$; since f is not defined at $x = 2$, that point cannot be a local extrema. Thus there are no local extrema of $f(x)$.
40. We must compare the values of the limits at the open endpoints and the value at the only interior local extrema $x = -\frac{1}{2}$ (note that $f(x)$ is defined on all of $(-1, 1)$, so there will be no vertical asymptotes in that interval). Since $\lim_{x\to -1^+} f(x) = -\frac{1}{2}$, $\lim_{x\to 1^-} f(x) = -\infty$, and $f(-\frac{1}{2}) = -\frac{1}{3}$, we can see that, on the interval $(-1, 1)$, $f(x)$ has a global maximum at $x = -\frac{1}{2}$ but no global minimum.
45. We already know there are no interior local extrema in the interval. However, $f(x)$ has a vertical asymptote at $x = 2$, which is in the interval $[1, 4]$. Thus we compare $\lim_{x\to 2} f(x) = \infty$, $f(1) = 1$, and $f(4) = \frac{1}{2}$ to see that, on the interval $[1, 4]$, $f(x)$ has a global minimum at $x = 4$ but no global maximum.
48. See the reading.
49. **Proof.** If $f(x)$ is a rational function with numerator of degree n and denominator of degree m, then its derivative $f'(x)$ is a rational function with numerator of degree $n + m - 1$ and denominator of degree $2m$ (see Problem 3). If $f(x)$ has a horizontal asymptote then $n \le m$, which implies that the degree of the numerator of $f'(x)$ is greater than the degree of the denominator of $f'(x)$, since $n + m - 1 \le m + m - 1 = 2m - 1 < 2m$. Therefore $f'(x)$ must have a horizontal asymptote.
52. See the reading.

Section 6.4

3. Roots, holes, and vertical and horizontal asymptotes (you describe how).
4. The intervals on which the function is increasing or decreasing and concave up or concave down, as well as any local extrema or inflection points (you describe how).
5. The location of any vertical asymptotes (and the behavior of the function near those asymptotes), the location of any horizontal asymptotes (and in general the behavior of the function at the "ends"), and any holes in the graph of the function (you describe how).
6. Part (a): Hint: Besides the "obvious" graph, try one where the coordinates of the hole are $(-1, -2)$ and another where $\lim_{x\to 2^+} f(x) = -\infty$.

   Part (b): Three possible answers are: $f(x) = \dfrac{-(x-1)(x-3)(x+1)}{(x-2)^2(x+1)}$, $f(x) = \dfrac{-(x-1)(x-3)(x+1)^2}{(x-2)^2(x+1)^2}$, and $f(x) = \dfrac{-(x-1)^2(x-3)(x+1)}{(x-2)(x+1)(x^2+1)}$.
7. Hint: The graph can cross its slant asymptote.
8. Hint: The graph can cross its horizontal asymptote.
9. Graph $y = \dfrac{(x-2)(x-5)(x^2+1)}{(x+1)(x-3)(x-5)}$ on your calculator. The graph of the function described by the number lines is like this graph (note there is a hole at $x = 5$). It helps to look at each subinterval individually before you try to draw the graph; that will help you recognize that there must be a hole, and not an asymptote, at $x = 5$.
16. After factoring you can see that the graph of $f(x)$ has no holes, roots at $x = 0$, $x = -3$, and $x = 1$, and vertical asymptotes at $x = \pm 2$. After polynomial long division you can see that the graph has a slant asymptote at $y = 2x + 4$. Plotting a few values or doing a sign analysis of $f(x)$ can help you fill in the rest of the graph (with number line marked at $x = -3$, $x = -2$ (DNE), $x = 0$, $x = 1$, and $x = 2$ (DNE) we have signs $-$, $+$, $-$, $+$, $-$, $+$).

Section 6.4 (continued)

17. The graph has a hole at $x = -1$, roots at $x = -\frac{3}{2}$ and $x = 1$, a vertical asymptote at $x = 3$, and a slant asymptote with equation $y = 2x + 7$. Plotting points or doing a sign analysis of $f(x)$ can help you fill in the rest of the graph.

23. $f(x) = \dfrac{3x^2 - 2x + 4}{(x-2)^2}$ (the numerator is an irreducible quadratic), so the graph has no roots or holes, a vertical asymptote at $x = 2$, and a horizontal asymptote at $y = 3$; moreover, the graph of $f(x)$ is always positive. $f'(x) = \dfrac{-2(5x+2)(x-2)}{(x-2)^4}$, so the number line for $f'(x)$ is marked at $x = -\frac{2}{5}$ and $x = 2$ (DNE) with signs $-$, $+$, $-$; thus $f(x)$ has a local minimum at $x = -\frac{2}{5}$. $f''(x) = \dfrac{4(5x+8)(x-2)^4}{(x-2)^8}$, so the number line for $f''(x)$ is marked at $x = -\frac{8}{5}$ and $x = 2$ (DNE) with signs $-$, $+$, $+$; thus $f(x)$ has an inflection point at $x = -\frac{8}{5}$. Finally, $f(-\frac{2}{5}) \approx 0.917$, $f(-\frac{8}{5}) \approx 1.148$, and $\lim\limits_{x\to 2} f(x) = \infty$ (from the right or left). With all this information you should be able to draw an extremely accurate graph of $f(x)$. Note that the graph approaches its horizontal asymptote of $y = 3$ from *below* as $x \to \infty$.

24. The graph of $f(x)$ has a hole at $x = -1$, roots at $x = 0$ and $x = 1$, a vertical asymptote at $x = 2$, and a slant asymptote with equation $y = x$. The number line for f is marked at $x = -1$ (DNE), $x = 0$, $x = 1$, and $x = 2$ (DNE) with signs $-$, $+$, $-$, $+$. $f'(x) = \dfrac{(x^2-4x+2)(x+1)^2}{(x-2)^2(x+1)^2}$ where the quadratic in the numerator is *not* irreducible, and by the quadratic formula has roots at $x = 2 \pm \sqrt{2}$. The number line for $f'(x)$ is marked at $x = -1$ (DNE), $x = 2 - \sqrt{2} \approx 0.586$, $x = 2$ (DNE), and $x = 2 + \sqrt{2} \approx 3.414$ with signs $+$, $+$, $-$, $-$, $+$; thus $f(x)$ has a local maximum at $x = 2 - \sqrt{2}$ and a local minimum at $x = 2 + \sqrt{2}$. $f''(x) = \dfrac{4(x-2)(x+1)^4}{(x-2)^4(x+1)^4}$ has a number line marked at $x = -1$ (DNE) and $x = 2$ (DNE) with signs $-$, $-$, $+$; thus $f(x)$ has no inflection points. Finally, $f(2-\sqrt{2}) \approx 0.172$, $f(2+\sqrt{2}) \approx 5.828$, $\lim\limits_{x\to 2^-} f(x) = -\infty$, and $\lim\limits_{x\to 2^+} f(x) = \infty$. With all this information you should be able to draw the graph of $f(x)$ (don't forget the hole at $x = -1$!).

31. $f(x) = \dfrac{-(x^2+1)(x-4)}{(x+2)(x-2)(x-4)}$.

32. $f(x) = \dfrac{A(x-2)(x+2)(x-6)}{(x-3)(x-6)}$ for some constant A; solve $\dfrac{A(6-2)(6+2)}{(6-3)} = 32$ to get $A = 3$.

33. $f(x) = \dfrac{A(x-1)(x+1)}{(x^2+1)^2}$ for some constant A; use the fact that $f(0) = 4$ to get $A = -4$.

35. $f(x)$ is clearly a quadratic with two holes in it. Suppose the quadratic part (*i.e.* the function f after reducing) is $g(x) = x^2 + bx + c$. Since $g(-4) = 2$ and $g(2) = 8$ we can solve a system of equations to find $b = 3$ and $c = -2$. Therefore $f(x) = \dfrac{(x^2+3x-2)(x+4)(x-2)}{(x+4)(x-2)}$.

37. This one is tricky. There is a slant asymptote that passes through the points $(0,1)$ and $(3,4)$, *i.e.* a slant asymptote with equation $y = x + 1$. We also know that $f(x)$ has a vertical asymptote at $x = 1$, so try $f(x) = (x+1) + \dfrac{A}{x-1}$ (this would be the form of the improper rational function $f(x)$ after polynomial long division). Use the fact that $f(0) = 2$ to solve for A (you'll find that $A = -1$). Thus $f(x) = (x+1) + \dfrac{-1}{x-1} = \dfrac{x^2-2}{x-1}$ might work; a check with a graphing calculator (including a graph of the slant asymptote) shows that it does work.