Homework for Week 15 **Math 231 Fall 2001**

This homework will not be collected. It is your responsibility to do as many problems as necessary to understand the material (this includes doing extra problems if you need more practice). We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 7.1 3, 4, 5, 8, 9, 14, 19, 25, 28, 31, 35, 42, 40, 47, 49, 56, 57, 58, 69, 71, 73.

Section 7.2 6, 7, 9, 11, 12, 13, 15, 20, 22, 23, 28, 30, 34, 35, 37, 40, 45, 46, 51, 53, 58, 63, 67, 69, 76, 82, 85, 87, 89.

Section 7.5 4, 11, 12, 17, 20, 23, 24, 28, 31, 35, 37, 41, 44, 46, 51, 57, 59, 64, 70, 71, 74, 75, 76.

Selected Hints and Answers

Caution: The hints and answers below are not necessarily full solutions. Many of them would not be considered complete on a quiz or test.

Section 7.1

5. $|x| = \sqrt{x^2}$.
8. True.
9. False.
14. Solve $3 - x > 0$ and $x^2 - 3x + 4 \neq 0$ to get $[3, 4) \cup (4, \infty)$.
19. Solve $\dfrac{x^2 - 1}{x^3 - 7x + 6} = \dfrac{(x-1)(x+1)}{(x-1)(x+3)(x-2)} \geq 0$ to get $(-3, 1] \cup (2, \infty)$.
25. 0 (split into two terms).
28. TYPO: Should be $x \to 0^+$. Answer is $-\infty$ (factor out $x^{-\frac{1}{3}}$, or $x^{-\frac{1}{2}}$).
31. $-\infty$ (factor out $x^{\frac{1}{3}}$).
35. $-\frac{1}{8}$ (combine fractions and multiply by conjugate).
40. $f'(x) = \dfrac{-(1 - 5x^{-\frac{1}{2}})}{(x - 10x^{\frac{1}{2}} - 25)^2}$ (write as quotient and multiply out first).
42. $f'(x) = \frac{7}{3}x^{\frac{4}{3}} + \frac{1}{6}x^{-\frac{7}{6}}$ (multiply out first).
47. $f'(x) = \dfrac{3(-x^{-\frac{1}{2}})(1 + x^{-\frac{1}{2}}) - 3(1 - 2\sqrt{x})(-\frac{1}{2}x^{-\frac{3}{2}})}{(1 + x^{-\frac{1}{2}})^2}$ (note use of constant multiple rule).
49. $f''(x) = \dfrac{\frac{1}{4}x^{-\frac{3}{2}}(1 + 2x^{\frac{1}{2}} + x) - \frac{1}{2}x^{-\frac{1}{2}}(x^{-\frac{1}{2}} + 1)}{(1 + 2x^{\frac{1}{2}} + x)^2}$.
56. After factoring, $f'(x) = \dfrac{(6x^{\frac{1}{4}} - 1)(3x^{\frac{1}{4}} - 1)}{2x^{\frac{1}{2}}}$; the CPs of f are $x = \frac{1}{\sqrt[4]{6}}$, $x = \frac{1}{\sqrt[4]{3}}$, and $x = 0$.
57. The only critical point is $x = 0$ ($f'(x)$ is never zero).
58. After factoring, $f'(x) = \dfrac{x(\frac{3}{2}x^{\frac{1}{2}} - 2)}{(x^{\frac{1}{2}} - 1)^2}$; the CPs of f are $x = 1$, $x = 0$, and $x = \frac{16}{9}$.
69. $f'(4) = \frac{1}{2}(4)^{-\frac{3}{2}} - \frac{1}{12}(4)^{-\frac{11}{12}} \approx -0.08588$ (simplify f first).
71. $f'(2) = \lim\limits_{h \to 0} \dfrac{\frac{1}{\sqrt{5(2+h)-(2+h)^2}} - \frac{1}{\sqrt{6}}}{h} = \ldots(\text{work})\ldots = -\dfrac{1}{12\sqrt{6}}$.

Section 7.1 (continued)

73. $f'(x) = \lim_{h \to 0} \frac{((x+h)^2 - 4)^{-\frac{1}{2}} - (x^2-4)^{-\frac{1}{2}}}{h} = \ldots\text{(work)}\ldots$
$= \lim_{h \to 0} \frac{-2x - h}{\sqrt{x^2-4}\sqrt{(x+h)^2-4}(\sqrt{x^2-4} + \sqrt{(x+h)^2-4})} = \ldots\text{(work)}\ldots \frac{-x}{(x^2-4)^{\frac{3}{2}}}.$

Section 7.2

7. Almost any example works; one is: $f(x) = x^2$, $g(x) = x + 1$.
9. Hint: Use $f(x) = (3x + \sqrt{x})(3x + \sqrt{x})$ for the second way, and $f(x) = 9x^2 + 6x^{\frac{3}{2}} + x$ for the third.
11. Hint: Use $f(x) = \sqrt{x}(1 - x^7)^{-1}$ for the second way.
12. $f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.
13. TYPO: $f(u)$ should say $f(x)$.
$\frac{(2\sqrt{3x^2+1}+15(\sqrt{3x^2+1})^4)(1-\sqrt{3x^2+1})-((\sqrt{3x^2+1})^2+3(\sqrt{3x^2+1})^5)(-1)}{(1-\sqrt{3x^2+1})^2}(\frac{1}{2}(3x^2+1)^{-\frac{1}{2}}(6x))$.
15. The quotient rule is first. Also need chain, sum, constant multiple, and power rules.
20. The sum rule is first. Also need chain, sum, constant multiple, and power rules.
22. $f'(x^{-2})(-2x^{-3})$.
23. $-2(f(x))^{-3}f'(x)$.
28. $f'(g(x)h(x))(g'(x)h(x) + g(x)h'(x))$.
30. $2xf(g(x)) + x^2 f'(g(x))g'(x)$.
34. 1.
35. -2.
37. 6.
40. 4.
45. True.
46. False.
51. $f'(x) = \frac{d}{dx}(x^2 + 3x - 1) \cdot (x-2)^{\frac{3}{2}} + (x^2 + 3x - 1) \cdot \frac{d}{dx}((x-2)^{\frac{3}{2}})$ (product rule)
$= (\frac{d}{dx}(x^2) + 3\frac{d}{dx}(x) - \frac{d}{dx}(1)) \cdot (x-2)^{\frac{3}{2}} + (x^2 + 3x - 1) \cdot \frac{3}{2}(x-2)^{\frac{1}{2}} \frac{d}{dx}(x-2)$
(sum and CM; chain and power)
$= (2x+3)(x-2)^{\frac{3}{2}} + (x^2 + 3x - 1)(\frac{3}{2})(x-2)^{\frac{1}{2}}(1)$. (power, sum)
58. $f'(x) = 3(-\frac{2}{3})((x^2+1)^8 - 7x)^{-\frac{5}{3}}(8(x^2+1)^7(2x) - 7)$.
69. $f(x) = x^{1+2+\frac{1}{2}+\frac{2}{3}} = x^{\frac{25}{6}}$, so $f'(x) = \frac{25}{6}x^{\frac{21}{6}}$.
76. $f''(2) = \frac{3}{4}$ (simplify $f(x)$ first!).
82. $x = \frac{1}{2}$, $x = 0$, $x = 1$.
85. Part (a): $\frac{dA}{dr} = 2\pi r$.
Part (b): no; yes.
Part (c): $\frac{dA}{dt} = \frac{d}{dt}(\pi(r(t))^2) = 2\pi r(t)r'(t) = 2\pi r\frac{dr}{dt}$.
Part (d): yes; yes.
Part (e): $\frac{dA}{dt}|_{r=24} = 2\pi(24)(2) = 96\pi$.
87. **Proof.** $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{d}{dx}(f(x)(g(x))^{-1})$ (algebra)
$= \frac{d}{dx}(f(x)) \cdot (g(x))^{-1} + f(x) \cdot \frac{d}{dx}((g(x))^{-1})$ (product rule)
$= f'(x) \cdot (g(x))^{-1} + f(x) \cdot (-g(x))^{-2}g'(x)$ (power, chain rules)
$= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{(g(x))^2} = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$. (algebra) ■

Section 7.2 (continued)

89. **Proof.** $\frac{d}{dx}(f(u(v(x)))) = f'(u(v(x))) \cdot \frac{d}{dx}(u(v(x)))$ (chain rule)

$= f'(u(v(x))) \cdot u'(v(x)) \cdot v'(x).$ (chain rule) ■

Similarly, $\frac{d}{dx}(f(u(v(w(x))))) = f'(u(v(w(x)))) \cdot u'(v(w(x))) \cdot v'(w(x)) \cdot w'(x)$.

Section 7.5

20. Solve $x^2 - 4x + 3 > 0$ to find domain $(-\infty, 1) \cup (3, \infty)$.

23. $f(x) = \dfrac{(x+1)^{\frac{2}{3}}}{(x-1)^{\frac{4}{3}}}$, so must be continuous everywhere except at $x = 1$.

24. $f'(x) = \dfrac{1}{3x^{\frac{1}{3}}\sqrt{1 + x^{\frac{2}{3}}}}$, which is defined everywhere but at $x = 0$, so f is differentiable everywhere except at $x = 0$.

28. $\lim_{x\to\infty} f(x) = 1 - \dfrac{3}{3} = 0$ (factor out $\sqrt{x^2} = x$ from the denominator), so f has a horizontal asymptote at $y = 0$ on the right. $\lim_{x\to-\infty} f(x) = 1 + \dfrac{3}{3} = 2$ (this time $\sqrt{x^2} = -x$, since x is negative), so f has a horizontal asymptote at $y = 2$ on the left. Also, $f(x) = 1 - \dfrac{3x}{\sqrt{(3x+1)^2}}$, so f has a vertical asymptote at $x = -\frac{1}{3}$ (since $\lim_{x\to-\frac{1}{3}} f(x)$ is infinite).

31. $f(x) = \dfrac{(x+3)(x-1)}{\sqrt{x}}$, so has roots at $x = -3$ and $x = 1$.

35. $f(x) = (-x^2+5)^{\frac{2}{3}}(x-1)^{\frac{2}{3}}$ can only change sign at $x = \pm\sqrt{5}$ and $x = 1$. Using a number line for f we find that f is positive on $(-\infty, -\sqrt{5}) \cup (1, \sqrt{5})$ and negative on $(-\sqrt{5}, 1) \cup (\sqrt{5}, \infty)$.

37. $f'(x) = \dfrac{-(9x^2+4)}{(x^2-4)^2\sqrt{x^2+1}}$ can only change sign at $x = \pm 2$. Using a number line for f' we find that f is always decreasing.

41. $f''(x) = \dfrac{4(x-3)}{3(x-1)^{\frac{5}{3}}}$ can only change sign at $x = 1$ and $x = 3$. Using a number line for f'' we find that f is concave up on $(-\infty, 1) \cup (3, \infty)$ and concave down on $(1, 3)$.

44. $f'(x) = \dfrac{15(x^2-1)}{\sqrt{1+x}}$, so the critical points of f are $x = 1$ and $x = -1$. Using the first derivative test we see that $x = 1$ is the only local extrema of f (a local minimum).

46. $f''(x) = \dfrac{4(2x^2-1)}{(x^2+1)^{\frac{5}{2}}}$ can only change sign at $x = \pm\frac{1}{\sqrt{2}}$. By looking at a number line for f'' we see that f has an inflection point at both $x = \frac{1}{\sqrt{2}}$ and $x = -\frac{1}{\sqrt{2}}$.

51. $f(x) = \dfrac{1}{\sqrt{(x-3)(x-1)}}$ is only defined on $(-\infty, 1) \cup (3, \infty)$, so we'll have to be careful between $x = 1$ and $x = 3$. $f'(x) = \dfrac{2-x}{((x-3)(x-1))^{\frac{3}{2}}}$, so only $x = 2$, $x = 1$, and $x = 3$ could be critical points of f; but the function f is not defined at any of these points. By comparing $f(0) \approx 0.577$, $f(10) \approx 0.126$, $\lim_{x\to1^-} f(x) = \infty$, and $\lim_{x\to3^+} f(x) = \infty$, we can see that f has no global maximum on $[0, 10]$, but has a global minimum on $[0, 10]$ at $x = 10$.

57. $f(x) = 2(x+1)^{\frac{3}{2}}(3x-7)$ has domain $[-1, \infty)$, and can only change sign at $x = \frac{7}{3}$; its number line (on $[-1, \infty)$, as with all number lines in this problem) reads $-$, $+$. On the interval $[-1, \infty)$, $f'(x) = 15(x-1)\sqrt{x+1}$ can only change sign at $x = 1$, and has a number line with $-$, $+$ (see Problem 44). $f''(x) = \frac{15(3x+1)}{2\sqrt{x+1}}$ can only change sign at $x = -\frac{1}{3}$ (on $[-1, \infty)$), and has a number line with $-$, $+$. We also have $f(-1) = 0$, $f(\frac{7}{3}) = 0$, $f(-\frac{1}{3}) \approx -8.709$, and $\lim_{x\to\infty} f(x) = \infty$. Check your graph with a graphing window of $-2 \le x \le 4$, $-30 \le y \le 40$.

Section 7.5 (continued)

61. $f(x)$ can only change sign at its zeros $x = -3$ and $x = 1$, and has a number line with $+$, $-$, $+$. $f'(x) = \dfrac{4x}{(x-1)^{\frac{2}{3}}}$ can only change sign at $x = 0$ and $x = 1$ (where it DNE), and has a number line with $-$, $+$, $+$. See Problem 41 for the concavity information. Note that at $x = 1$ the function exists but the derivative does not. Thus we compute $\lim_{x\to 1^-} f'(x) = \infty$ and $\lim_{x\to 1^+} f'(x) = \infty$, and find that f has a vertical tangent line at $x = 1$. We also have $f(-3) = 0$, $f(0) = -9$, $f(1) = 0$, $\lim_{x\to\infty} f(x) = \infty$, and $\lim_{x\to-\infty} f(x) = \infty$. Try checking your answer with a graphing window of $-5 \le x \le 5$, $-10 \le y \le 40$ (the concavity is difficult to see from the graph).

64. The constraint is $\pi r^2 h = 40$; want to find the minimum of $SA = 2\pi r^2 + 2\pi r h$, so minimize $SA(r) = 2\pi r^2 + \frac{80}{r}$ on the interval $r \in (0, \infty)$. The only critical point is $r = \sqrt[3]{\frac{20}{\pi}} \approx 3.435$, and it is a local minimum by the first derivative test. Since $\lim_{r\to 0} SA(r) = \infty$ and $\lim_{r\to\infty} SA(r) = \infty$, while $SA(3.435) \approx 44.87$, the critical point is a *global* minimum. With this value of r we have $h \approx 3.707$. Thus the oil drums should be constructed so they are 3.707 feet high with a radius of 3.435 feet.

70. Minimize $C(x) = 3x + 5\sqrt{(800-x)^2 + 500^2}$ on the interval $(0, 800)$. This function has two critical points, but only one ($x = 425$) is in the desired interval. This critical point is a local minimum with value $C(425) = 4400$, while $\lim_{x\to 0} C(x) \approx 4717$ and $\lim_{x\to 800} C(x) = 4900$; thus $x = 425$ is a *global* minimum. The steam pipe should be buried along the 800 foot side of the parking lot for 425 feet, and then diagonally under the parking lot to the opposite corner.

71. One way to do this gives a constraint of $(x-4)(y-2) = 20$ and a printed area of $A = xy$. With this setup, we must minimize $A(x) = \frac{20x}{x-4} + 2x$ on the interval $(4, \infty)$. On this interval, $A(x)$ has only one critical point, $x = 4 + 2\sqrt{10} \approx 10.32$; this critical point is a local minimum by the first derivative test. Since $A(10.32) \approx 53.3$ while $\lim_{x\to 4^+} A(x) = \infty$ and $\lim_{x\to\infty} A(x) = \infty$, this critical point is the *global* minimum on the interval. Thus Alina should make fliers with paper that is 10.32 inches wide and 5.16 inches high.

74. Hint: Show that f is defined at $x = 3$, but that f' is not; then show that $\lim_{x\to 3^+} f'(x) = \infty$ while $\lim_{x\to 3^-} f'(x) = -\infty$ to conclude that f has a vertical cusp (point down) at $x = 3$. (Note: The fact that the cusp is vertical is difficult to see from a calculator graph unless you zoom in a lot.)

75. TYPO: Should say $x = 2$.
Hint: Show that f is defined at $x = 2$, but that f' is not; then show that $\lim_{x\to 2^+} f'(x) = \infty$ and $\lim_{x\to 2^-} f'(x) = \infty$ to conclude that f has a vertical tangent line at $x = 2$.

76. Hint: Show that $\lim_{x\to -2} f(x)$ exists (it is equal to $(-4)^{\frac{2}{5}} \approx 1.7411$), but is not equal to $f(-2)$ (which does not exist); conclude that the graph of f has a "hole" at $x = -2$.