Homework for Week 2 **Math 231 Fall 2001**

This homework will not be collected; it is your responsibility to do as many problems as necessary to understand the material. We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 0.5 2, 3, 4, 5, 7, 9, 12, 13, 16, 17, 19, 20, 22, 23, 24, 27, 28, 31, 34, 44, 45, 46.

Section 1.1 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19.

Section 1.2 2, 3, 4, 5, 7, 9, 10, 11, 12, 13, 15, 16, 18, 19, 23, 25, 26, 28, 29, 30, 31, 33, 34, 35, 37.

Section 1.3 1, 2, 3, 5, 6, 8, 10, 12, 16, 17, 18, 20, 21, 22, 24, 26, 28, 29, 30, 31, 32, 34, 36, 37, 38, 39, 41, 42, 44, 46.

Selected Hints and Answers

Caution: The hints and answers below are not full solutions. Most of them would not be considered complete on a quiz or test.

Section 0.5

2. Hint: See the "Caution" after Example 1.
7. All positive integers greater than or equal to 4.
12. Hint: When $n = 4$, $2n = 8$; thus $2 + 4 + 6 + \ldots + 2n = 2 + 4 + 6 + 8 = 20$.
16. $2 + 4 + 6 + \ldots + 2(k-1) = (k-1)((k-1)+1)$.
19. First note that the statement (Not C) $\Rightarrow B$ is logically equivalent to its contrapositive (Not B) $\Rightarrow C$; it is the contrapositive that is used in the proof below.
    Proof: A is true by hypothesis.
    Thus (Not B) is true. (since $A \Rightarrow$ (Not B))
    Therefore C must be true. (since (Not B) $\Rightarrow C$) ■
22. Part (b): To use proof by contradiction to show that A is true, we must assume that A is *false* and show that that assumption leads to a logical contradiction.
    Proof: Seeking a contradiction, suppose A is false.
    Since we are given that (Not A) $\Rightarrow B$, this means that B is true.
    Since $C \Rightarrow$ (Not B), we know that $B \Rightarrow$ (Not C), and thus C is false.
    However, we are given that C is true.
    Since C cannot be both true and false, we have a contradiction.
    Therefore our original assumption that A is false must itself be false.
    Therefore A must be true. ■
23. Hint: Show that if r, s, t, and u are integers, then $\frac{r}{s} + \frac{t}{u}$ can be written as the quotient of two integers.
24. Hint: Use proof by contradiction and the result in Problem 23.
28. Hint: If $M = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$, use the distance formula to show that $\text{dist}(M,P) = \text{dist}(M,Q)$.
31. Part (a): This fact is easy to prove with a simple calculation; the only tricky part is writing down your argument like a proof.
    Proof: If a, b, and c are any real numbers with $b \neq 0$ and $c \neq 0$, then:
    $$\frac{a/b}{c} = \frac{a/b}{c/1} \text{ (since } c = c/1) = \frac{a(1)}{bc} \text{ (by the given "fact")} = \frac{a}{bc}. \quad ■$$

Section 0.5 (continued)

45. **Proof**: (by induction)

True for $n = 1$: The statement is true for $n = 1$, since $2 = 1(1 + 1)$.

Inductive hypothesis: Assume true for $n - 1$, *i.e.* assume that:

$$2 + 4 + 6 + \ldots + 2(n - 1) = (n - 1)((n - 1) + 1) = (n - 1)(n).$$

Prove true for n given inductive hypothesis:

$$\begin{aligned} 2 + 4 + 6 + \ldots + 2n &= 2 + 4 + 6 + \ldots + 2(n - 1) + 2n && (1) \\ &= (2 + 4 + 6 + \ldots + 2(n - 1)) + 2n && (2) \\ &= (n - 1)n + 2n && (3) \\ &= n^2 - n + 2n = n^2 + n = n(n + 1). && (4) \end{aligned}$$

Step (1) is just writing out the second-to-last term; Step (2) is grouping; Step (3) uses the inductive hypothesis; and Step (4) is algebra. ■

Section 1.1

1. There are many possible answers to these problems. We give one possible answer for each part.

Part (a): f(person in the U.S.) = the state that person is in right now. This is a function because *each* person in the U.S. is in *exactly one* state right now. The range of this particular function is the set of *all* U.S. states, since every state has at least one person in it right now.

Part (b): f(person in the U.S.) = the states that person has visited this year. This is *not* a function because some people may have visited more than one state this year; there is not a *unique* output for every possible input.

Part (c): The function f(person in the U.S.)=Kansas is a constant function because *every* person in the U.S. is assigned to the *same* state (Kansas) under the rule defined by f.

Part (d): There is no identity function from the set of all people in the U.S. to the set of all U.S. states, because the domain and the target are different sets. An identity function should send each input to *itself* as an output, and that is not possible here.

5. Again, there are many possible answers, and we give one possible answer for each part. We will do only the tables, and leave the "diagrams" to you. Let elements of the set B be represented by the variable x.

Part (a): One possible function $f : B \to A$ is given in the table below:

x	1	2	3	4
$f(x)$	6	2	2	10

The range of the function above is $\{2, 6, 10\}$.

Part (b): The table below does *not* define a function from B to A:

x	1	2	3
$f(x)$	6	2	8

The table above does not define $f(x)$ for *each* value of $B = \{1, 2, 3, 4\}$. Thus although it *is* a function on the set $\{1, 2, 3\}$, it is *not* a function $B \to A$.

Part (c): The table below does *not* define a function from B to A:

x	1	2	3	2
$f(x)$	6	10	8	4

The table above gives *two* outputs (namely 10 and 4) for the input $x = 2$, and thus does not define a function $B \Rightarrow A$.

Section 1.1 (continued)

9. Hints: Are there any inputs that will produce the output $y = 5$? If so, what are they? If not, why not? What about the output $y = 0$? For the last part, show that you can always solve $y = x^2 + 1$ for x if y is in the set $[1, \infty)$.
10. The domain is the solution to the equation $(x-1)(2x+3) \geq 0$, which is $(-\infty, -\frac{3}{2}) \cup (1, \infty)$.
12. $(0, \frac{5}{3}) \cup (\frac{5}{3}, \infty)$.
13. Part (c): $f(a+1) = \frac{a+1}{1-(a+1)} = -\frac{a+1}{a}$.
    Part (d): $f(x^2+x) = \frac{x^2+x}{1-(x^2+x)} = \frac{x^2+x}{1-x^2-x}$.
15. Let l be the number of shoes that Linda has (the dependent variable), and let a be the number of shoes that Alina has (the independent variable). We can write l as a function of a as $l(a) = a + 3$.

Section 1.2

2. $\{(x, \sqrt{x}) \mid x \geq 0\}$ is one way to represent the graph of $f(x) = \sqrt{x}$ in set notation. We could also write $\{(x, y) \mid x \geq 0 \text{ and } y = \sqrt{x}\}$.
7. Part (a): y-intercept at $y = 2$, no roots.
   Part (b): positive on $(-\infty, \infty)$.
   Part (c): increasing on $(-\infty, 0)$, decreasing on $(0, \infty)$.
   Part (d): concave up on approximately $(-\infty, -1) \cup (1, \infty)$, concave down on approximately $(-1, 1)$.
   Part (e): Local maximum at $x = 0$ (with a value of $f(0) = 2$). This is also the global maximum. There is no global minimum.
   Part (f): Inflection points at approximately $x = -1$ and $x = 1$.
   Part (g): Horizontal asymptote at $y = 1$.
18. Verbal Description: The graph of f is below or on the x-axis on all of I. Mathematical Description: For all $x \in I$, $f(x) \leq 0$.
23. There are three local extrema; we will only discuss the local maximum at $x = 2$. For this local extrema, $\delta = 1$ will work: If $x \in (c-\delta, c+\delta) = (2-1, 2+1) = (1, 3)$, then $f(2) \geq f(x)$ (look at the graph on $(2, 3)$). The *largest* possible δ that seems to work is approximately $\delta = 1.5$, since $f(2)$ is greater than or equal to $f(x)$ for all x in $(2-1.5, 2+1.5) = (0.5, 3.5)$, but $f(x)$ is larger than $f(2)$ for any values of x that are further to the left (*e.g.* $f(0)$ is larger than $f(2)$).
25. One good window for this graph is $x \in [-1, 1]$, $y \in [-0.3, 1]$.
28. $x = -1$ and $x = 3$.
31. Hint: $f(x) = x^2 - 2x + 3 = (x+1)(x-3)$ is positive when both $x+1$ and $x-3$ are positive, and also when both $x+1$ and $x-3$ are negative. Solve this set of inequalities to find the intervales on which f is positive. A similar set of inequalities can be used to find the intervals on which f is negative.
34. TYPO: THE PROBLEM SHOULD SAY "DECREASING."
    We must show that for a and b in $(-\infty, 3)$, $a < b$ implies $f(a) > f(b)$. We can do this as follows:
    Suppose $a < b$ are in $(-\infty, 3)$. Then $a - 3 < b - 3$. Note that both $a - 3$ and $b - 3$ have the same sign (why?); thus the previous inequality implies that $\frac{1}{a-3} > \frac{1}{b-3}$, and thus $f(a) > f(b)$.

Section 1.3

3. Since the slope is equal to $\frac{\Delta y}{\Delta x}$, we have $\frac{\Delta y}{4} = -3$, so $\Delta y = -12$.

Section 1.3 (continued)

8. Hint: Does the graph of a proportional function have to pass through the origin? Also, the function $f(x)$ can only be linear if its average rate of change is the same on every interval; find the average rate of change on $[0,1]$ and $[1,4]$.
10. The missing entries in the first row are 4 and 9. The missing entries in the second row are -17 and -26.
16. Slope $\frac{1}{6}$ and y-intercept $\frac{2}{3}$.
21. The average rate of change of $f(x) = x^2 - 10$ on $[0,2]$ is equal to 2.

    (Note that you can use graphs to see if your answers in this block are reasonable.)
24. The obvious way is to write $y - 1 = 3(x - 0)$. The point-slope form of a line is not unique. In this example we can write $y - y_0 = 3(x - x_0)$ for any point (x_0, y_0) on the graph; for example, since $(1,4)$ is on the graph (why?), we have $y - 4 = 3(x - 1)$.
29. $y = -x - 1$.

    (Note that you can check the answers in this block with a graphing calculator.)
31. $y = 2x - 2$.
34. Let h be the number of hours each week spent watching television (this is the dependent variable), and let d represent the amount (in dollars) of credit card debt (this is the independent variable). Then h is proportional to d, so $h(d) = kd$ for some constant k.
37. Let t be the number of years after 1985 (this is the independent variable), and let P represent the number of Whimsians on the island (this is the dependent variable). It is given that P is a linear function of t, and that $P(0) = 35$ and $P(15) = 91$. Thus $P(t) = \frac{56}{15}x + 35$.
39. Let t be the number of years after 1996 (the independent variable), and let W be the number of wombats on the island. $W(t)$ is a linear function with slope 4 that passes through the point $(5, 376)$. Therefore $W(t) = 4t + 356$, so there were originally $W(0) = 4(0) + 356 = 356$ wombats deserted on the island.
41. Hint: Once you find a linear function that you believe fits the data, you can use a graphing calculator to check that your function passes through the correct data points. You can also use this graph to verify your answer to the last part of the question.
46. Hint: Show that if $f(x) = -2x + 4$, then $\frac{f(b)-f(a)}{b-a} = 2$ for all real numbers a and b.