Homework for Week 4 **Math 231 Fall 2001**

This homework will not be collected; it is your responsibility to do as many problems as necessary to understand the material. We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 1.7 1, 2, 4, 6, 7, 8, 9, 10, 14, 16, 18, 22, 25, 26, 27, 29, 31, 32, 33, 35, 36, 37, 39, 40, 41, 42, 43, 45, 46, 49.

Section 2.1 2, 3, 4, 5, 6, 7, 8, 9, 12, 13, 14, 15, 16, 18, 20, 21, 22, 23, 25, 27, 28, 30, 31, 33, 35.

Section 2.2 1, 2, 3, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 19, 20, 21, 22, 24, 26, 27, 28, 31, 33, 34, 35, 36, 38, 39, 41.

Selected Hints and Answers

Caution: The hints and answers below are not full solutions. Most of them would not be considered complete on a quiz or test.

Section 1.7

4. Hint: Your goal is to find some values a and b so that $f(a) = f(b)$.
8. $(3, 2)$.
14. A function *can* be its own inverse (although this is rare). One example is the function $f(x) = \frac{1}{x}$; can you think of any others? The graph of a function that is its own inverse would not change if you reflected it over the line $y = x$ (why?).
18. Suppose there are some real numbers $a \geq 0$ and $b \geq 0$ so that $f(a) = f(b)$; we want to show that the only way this can happen is if $a = b$. Using the definition of $f(x)$ we have:
$$f(a) = f(b) \quad \Rightarrow \quad a^2 + 1 = b^2 + 1 \quad \Rightarrow \quad a^2 = b^2;$$
Finally, if $a^2 = b^2$ and a and b are nonnegative, then we must have $a = b$. Therefore, if $f(a) = f(b)$ (and a and b are nonnegative), then a must equal b. This proves that the function $f(x) = x^2 + 1$ is one-to-one on the restricted domain $[0, \infty)$.
22. Hint: You must show that $f(x) = 1 + 15x$ is increasing on its entire domain $(-\infty, \infty)$. Use the definition of what it means for a function to be increasing; in other words, show that if $a < b$ then $f(a) < f(b)$.
27. We must show that $f(g(x)) = x$ for all x in the domain of g and that $g(f(x)) = x$ for all x in the domain of f:
$$f(g(x)) = f(\tfrac{x}{1+x}) = \frac{\frac{x}{1+x}}{1 - \frac{x}{1+x}} = \frac{\frac{x}{1+x}}{\frac{(1+x)-x}{1+x}} = \frac{x}{(1+x) - x} = \frac{x}{1} = x.$$
The calculation above is true for all $x \neq -1$ (*i.e.* for all x in the domain of g). The second calculation is similar (yes, you must do it).
33. We will write y as a function of x, "switch variables," and solve for the (new) y in terms of the (new) x:
$$y = \tfrac{x-1}{x+1} \quad \xrightarrow{\text{switch}} \quad x = \tfrac{y-1}{y+1} \quad \Rightarrow \quad x(y+1) = y - 1 \quad \Rightarrow \quad xy + x = y - 1$$
$$\Rightarrow \quad xy - y = -x - 1 \quad \Rightarrow \quad y(x-1) = -x - 1 \quad \Rightarrow \quad y = \tfrac{-x-1}{x-1}.$$
Therefore $f^{-1}(x) = \frac{-x-1}{x-1}$. Note that alternatively, we could have solved the original equation for x and *then* "switched" the variables x and y.
39. Hint: The two "largest" restricted domains on which this function is invertible appear to be approximately $(-\infty, 1.2]$ and $[1.2, \infty)$.

Section 1.7 (continued)

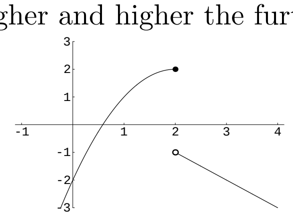
42. Part (a): $C(n) = 20 + 12n$.
Part (b): Suppose there were a and b in the domain $(-\infty, \infty)$ of $C(n)$ with $C(a) = C(b)$. In this case $20 + 12a = 20 + 12b$, so $12a = 12b$, and therefore we must have $a = b$. Therefore $C(n)$ is an invertible function.
Part (c): $n(C) = \frac{C-20}{12}$. Note that it does *not* make sense to "switch" the variables in this example, since both C and n are named to suggest particular quantities (C for "cost" and n for "number"). The function $n(C)$ describes the number n of shirts that you can have made for C dollars (the cost C is the input and the number n is the output).
Part (d): $n(150) = \frac{150-20}{12} = \frac{130}{12} \approx 10.83$, so you can make ten shirts for \$150 (why did we have to round *down* here?).

45. **Proof**: Suppose f is increasing on its entire domain. To show that f is one-to-one we will show that for all a and b in the domain of f, we have $a \neq b \Rightarrow f(a) \neq f(b)$, as follows:
 If $a \neq b$ then either $a < b$ or $a > b$.
 If $a < b$ then $f(a) < f(b)$ (since f is increasing), and thus $f(a) \neq f(b)$.
 If $a > b$ then $f(a) > f(b)$ (since f is increasing), and thus $f(a) \neq f(b)$.
In any case, if $a \neq b$ then $f(a) \neq f(b)$, and thus f is by definition a one-to-one function. The proof that a decreasing function is one-to-one is similar (can you write it down?). ■

46. Hint: Look at the proof of Theorem 3.

Section 2.1

5. As x approaches $-\infty$, $f(x)$ approaches ∞. In other words, as we travel along the graph of f towards the left, the graph eventually gets higher and higher the further left we go.

12. One possible graph with the given properties is 

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14. Since the left and right limits exist and are both equal to 5, the limit $\lim_{x\to 1} f(x)$ is also equal to 5. However, since the limit of $f(x)$ as $x \to 1$ does not concern the *value* of f at $x = 1$, you cannot say anything about the value of $f(1)$.

16. Hint: By the information given, you know one value that $\lim_{x\to 2^-} f(x)$ can *not* be; what is that value and why?

18. f has a horizontal asymptote at $y = 3$ and a vertical asymptote at $x = 1$.

20. Hint: Although the table seems to suggest that the limit of $f(x)$ as $x \to \infty$ is approximately 57, this is not necessarily the case. The table only tells us what the function is like at a few x-values at or before $x = 10,000$; the function could have a different behaviour after that. Draw a function whose graph goes through each of the datapoints in the table and then increases to ∞ sometime after $x = 10,000$.

21. Hint: The table seems to suggest that the limit of $f(x)$ as $x \to 2$ is approximately equal to 3. However, it is possible for a function f to seem like it is approaching 3 as $x \to 2$ and then at some point very close to $x = 2$ do something different. Draw such a function.

22. (a) -2; (b) 1; (c) -2; (d) 2; (e) 2; (f) -2; (g) 2; (h) does not exist.

28. Hint: Use your graphing calculator to graph $y = \frac{x-1}{x^2-2x+1}$ in an appropriate window, and then "trace" along the graph as $x \to 1$ from both sides. What is the height of the graph approaching as $x \to 1^-$ and as $x \to 1^+$? Although the function $y(x)$ is not defined at $x = 1$ (why?), it does have a well-defined limit as $x \to 1$.

30. Hint: Make a table of values of $x^2 + x + 1$ for a series of x-values that approach 2 from the left; for example, you might use $x = 1.9$, $x = 1.99$, $x = 1.999$, $x = 1.9999$, and $x = 1.99999$.

Section 2.1 (continued)

31. One possible table of approximate values is the one below:

x	2.9	2.99	2.999	3	3.001	3.01	3.1
$\frac{1}{x^2-x-3}$	-2.56	-25.06	-250.06	-	249.94	24.94	2.44

From the table above it appears that $\frac{1}{x^2-x-3} \to -\infty$ as $x \to 3^-$, and that $\frac{1}{x2-x-3} \to \infty$ a $x \to 3^+$. Thus we would approximate that $\lim\limits_{x\to 3^-} \frac{1}{x^2-x-3} = -\infty$ and that $\lim\limits_{x\to 3^+} \frac{1}{x^2-x-3} = \infty$; since the left and right limits are not equal, this would mean that $\lim\limits_{x\to 3} \frac{1}{x^2-x-3}$ does not exist.

33. Hint: Make a table of values of $3x+1$ for values of x that get progressively "smaller" (towards $-\infty$, so in other words, a sequence of x-values that are negative with larger and larger magnitudes). For example, you could use the values $x=-10$, $x=-100$, $x=-1,000$, $x=-10,000$, and $x=-100,000$.

Section 2.2

1. Statement (1): $\sqrt{x+7}$ approaches 2 as x approaches -3.
   Statement (2): We can make $\sqrt{x+7}$ as close as we like to 2 by making x sufficiently close (but not equal) to -3.
   Statement (3): For any small distance ϵ around 2, there is a small distance δ around -3 such that when x is within δ of -3, $\sqrt{x+7}$ is within ϵ of 2.
   Definition 1: For all $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x+3| < \delta \Rightarrow |\sqrt{x+7}-2| < \epsilon$.
   Statement (4): For all $\epsilon > 0$, there exists a $\delta > 0$ such that $x \in (c-\delta, c) \cup (c, c+\delta) \Rightarrow \sqrt{x+7} \in (L-\epsilon, L+\epsilon)$.
3. Hint: In this problem the letter "h" is playing the role of "x".
5. $0 < |x-5| < 0.01$ and $|f(x)+2| < 0.5$.
6. $x \in (1.9, 2) \cup (2, 2.1)$.
11. Hint: Try to write a statement of the form "$f(x) \in$ _______ ".
13. Hint: Sketch the graph of $y = x^2$ and show an ϵ-interval around $y = 4$ with $\epsilon = 0.4$ (*i.e.* show the horizontal "bar" between $y = 3.6$ and $y = 4.4$). Then draw the δ-interval around $x = -2$ with $\delta = 0.075$, excluding $x = -2$ (*i.e.* show the vertical "bar" between $x = -2.075$ and $x = -1.925$, excluding $x = -2$). Indicate on your graph that every x-value in $(-2.075, -2) \cup (-2, -1.925)$ has an $f(x)$-value in $(3.6, 4.4)$.
15. Using the calculator's "trace" feature on a graph of $y = x^2$ we can see that if $x \in (-2.5, -1.5)$ then $x^2 \in (2.25, 6.25)$. The interval $(2.25, 6.25)$ is not evenly centered around $y = 4$; and since $4 - 2.25 = 1.75$ and $6.25 - 4 = 2.25$, the largest "ϵ" we can guarantee around $y = 4$ is $\epsilon = 1.75$.
16. For all $\epsilon > 0$, there exists a $\delta > 0$ such that $x \in (3-\delta, 3) \Rightarrow |(4-x^2)+5| < \epsilon$.
19. For all $M > 0$, there exists a $\delta > 0$ such that $x \in (1, 1+\delta) \Rightarrow \frac{1}{1-x} > M$.
20. For all $\epsilon > 0$, there exists an $N > 0$ such that $x > N \Rightarrow |\frac{x}{1-2x} - 0.5| < \epsilon$.
21. For all $M > 0$, there exists an $N > 0$ such that $x > N \Rightarrow 3x - 1 > M$.
31. For all $\epsilon > 0$, there exists an $N < 0$ such that $x < N \Rightarrow |f(x) - L| < \epsilon$.
34. Using a calculator's "trace" feature we can estimate that every value of x in the interval $(1.957, 2.041)$ has an $f(x) = x^3$ value within 0.5 of $y = 8$. Since $2 - 1.957 = 0.043$ and $2.041 - 2 = 0.041$, the largest "working" δ (making a centered interval around $x = 2$ is $\delta = 0.041$.
38. $\delta \approx 0.0005$.
39. $N \approx 5.053$.