Homework for Week 5 **Math 231 Fall 2001**

This homework will not be collected; it is your responsibility to do as many problems as necessary to understand the material. We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 2.3 1, 3, 4, 5, 6, 7, 8, 9, 12, 13, 16, 19, 21, 22, 23, 24, 25, 26, 29, 33, 36, 38, 39.

Section 2.4 1, 2, 3, 6, 7, 9, 13, 14, 17, 19, 20, 25, 26, 27, 28, 30, 31, 32, 34, 35, 37, 38, 39, 41, 42, 43, 45, 47, 48, 49, 53.

Selected Hints and Answers

Caution: The hints and answers below are not full solutions. Most of them would not be considered complete on a quiz or test.

Section 2.3

3. For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x+2| < \delta \Rightarrow |\frac{3}{x+1} + 3| < \epsilon$.
4. The proof states that given any $\epsilon > 0$, a choice of $\delta = \epsilon/3$ should "work"; since we want $3x - 1$ to be within 0.25 of 5, we have $\epsilon = 0.25$ here, and thus we should choose $\delta = 0.25/3 \approx 0.083333$. Since our δ-interval is centered around $x = 3$, the punctured interval we are looking for is $(3-0.083333, 3) \cup (3, 3+0.08333)$, which equals $(2.91667, 3) \cup (3, 3.083333)$. To verify this with a graph, make a graph of $y = 3x - 1$ and verify that the height of the graph at each x-value in the punctured interval found above is within 0.25 of 5 (*i.e.* is in the interval $(4.75, 5.25)$). Draw vertical lines at $x = 2.91667$ and $x = 3.08333$ to make the "δ-bar," and draw horizontal lines at $y = 4.75$ and $y = 5.25$ to make the "ϵ-bar."
5. Hint: Don't forget that δ should be chosen as the *minimum* of 1 and 6ϵ.
6. Hint: Since this is a one-sided limit, your interval should only be to one side of $x = 0$.
7. Hint: This time the interval of x values is a set of *large* values for x rather than a punctured interval around some number; why?
12. $|(2x+4) - 6| = |2x - 2| = 2|x - 1|$, so if $|x - 1| < \delta$ then $|(2x+4) - 6| < 2\delta$. Thus we should choose $\delta = \epsilon/2$ (since we want $2\delta = \epsilon$).
16. $|(3x^2 + 1) - 1| = |3x^2| = 3|x|^2$, so if $|x| < \delta$ then $|(3x^2 + 1) - 1| < 3\delta^2$. We would like to have $3\delta^2 = \epsilon$, so choose $\delta = \sqrt{\epsilon/3}$.
19. $|(x^2 - 2x - 3) - 0| = |x^2 - 2x - 3| = |(x-3)(x+1)| = |x-3||x+1|$, so if $|x - 3| < \delta$ then $|(x^2 - 2x - 3) - 0| < \delta|x+1|$. We need to get a bound on the $|x+1|$ part of the expression. If we assume that $\delta < 1$ then $0 < |x-3| < \delta$ implies that $0 < |x-3| < 1$, and thus $x \in (2, 4)$ (but $x \neq 3$). The largest that $|x+1|$ can be while $x \in (2, 4)$ is $|4+1| = |5| = 5$. Thus if $\delta < 1$ we know that $|(x^2 - 2x - 3) - 0| < \delta(5)$. Therefore we should choose $\delta = \min(1, \epsilon/5)$.
21. If $x \in (-2, -2+\delta)$ then $-2 < x < -2+\delta$ and thus $0 < x+2 < \delta$. In this case $|(1 + \sqrt{x+2}) - 1| = |\sqrt{x+2}| = \sqrt{|x+2|} = \sqrt{x+2}$ (since $x + 2 > 0$), which is less than $\sqrt{\delta}$ (since $x + 2 < \delta$). Thus we should choose $\delta = \epsilon^2$ (since then $\sqrt{\delta} = \epsilon$). This has to do with the limit $\lim\limits_{x \to -2^+} (1 + \sqrt{x+2}) = 1$.
22. If $x \in (1-\delta, 1)$ then $1 - \delta < x < 1$, so $-\delta < x - 1 < 0$. We need to relate this fact somehow to the expression $\frac{1}{1-x}$. Since $-\delta < x - 1$ we have $\delta > 1 - x$, and since both sides of this inequality are positive we thus have $\frac{1}{1-x} > \frac{1}{\delta}$. Since we would like to have $M = \frac{1}{\delta}$ we should choose $\delta = \frac{1}{M}$. This has to do with the limit $\lim\limits_{x \to 1^-} \frac{1}{1-x} = \infty$.

Section 2.3 (continued)

23. $|\frac{x-1}{x} - 1| = |\frac{x-1-x}{x}| = |\frac{-1}{x}| = \frac{1}{|x|}$. Since $x > N$ we know that the quantity computed above is $< \frac{1}{N}$. We wanted to show it was less than ϵ, so we should choose $N = \frac{1}{\epsilon}$ (since then $\frac{1}{N} = \epsilon$). This has to do with the limit $\lim_{x\to\infty} \frac{x-1}{x} = 1$.

24. If $x > N$ then $x^2 + 2 > N^2 + 2$. Since we would like to have $M = N^2 + 2$ we should choose $N = \sqrt{M-2}$. This has to do with the limit $\lim_{x\to\infty}(x^2+2) = \infty$.

25. Hint: You must show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x-1| < \delta \Rightarrow |(2x+4) - 6| < \epsilon$. Your proof should look just like the proof in Example 3 in the reading. By problem 12 we know to choose $\delta = \epsilon/2$.

26. Hint: Again your proof should have exactly the same form as the one in Example 3 in the reading. You should find that δ should be chosen to be equal to ϵ.

29. We have to show that for all $\epsilon > 0$, there exists $\delta > 0$ so that $0 < |x| < \delta \Rightarrow$ $|(3x^2+1) - 1| < \epsilon$.
Proof: Given $\epsilon > 0$, choose $\delta = \underline{\sqrt{\epsilon/3}}$.
With this choice of δ, if $0 < |x| < \delta$ then:
$$|(3x^2+1) - 1| = |3x^2| = 3|x|^2 < 3\delta^2 = 3(\sqrt{\epsilon/3})^2 = 3(\epsilon/3) = \epsilon,$$
and thus $|(3x^2+1) - 1| < \epsilon$ for all $|x| < \delta$. ■

33. We have to show that for all $\epsilon > 0$, there exists a $\delta > 0$ such that $0 < |x-3| < \delta \Rightarrow$ $|(x^2 - 2x - 3) - 0| < \epsilon$. By problem 19 we know to choose $\delta = \min(1, \epsilon/5)$ (this choice will also be apparent about halfway through the proof).
Proof: Given $\epsilon > 0$, choose $\delta = \underline{\delta = \min(1, \epsilon/5)}$.
With this choice of δ, if $0 < |x-3| < \delta$ then:
$$|(x^2-2x-3) - 0| = |x^2-2x-3| = |(x-3)(x+1)| = |x-3||x+1| < \delta|x+1|.$$
Moreover, since we are assuming that $\delta < 1$ we know that $0 < |x-3| < 1$, and thus that $x \in (2,4)$ (but $x \neq 3$). Therefore the largest that $|x+1|$ could be is $|x+4| = |5| = 5$. Thus:
$$|(x^2-2x-3) - 0| = \cdots < \delta|x+1| < \delta(5) = (\epsilon/5)(5) = \epsilon,$$
and thus $|(x^2-2x-3) - 0| < \epsilon$ for all $0 < |x-3| < \delta$. ■

36. Hint: Problem 21 shows all the relevant calculations. Your proof should have the same general structure as the proof in Example 8 from the reading.

38. We have to show that for all $M > 0$, there exists a $\delta > 0$ such that $x \in (-2, -2+\delta) \Rightarrow$ $\frac{1}{x+2} > M$.
Proof: Given $M > 0$, choose $\delta = \underline{1/M}$.
If $x \in (-2, -2+\delta)$ then $-2 < x < -2+\delta$, and thus $0 < x+2 < \delta$. Therefore for our choice of δ we have:
$$\frac{1}{x+2} > \frac{1}{\delta} = \frac{1}{1/M} = M.$$
Therefore for all $x \in (-2, -2+\delta)$ we have $\frac{1}{x+2} > M$. ■

39. We need to show that for all $\epsilon > 0$, there exists an $N > 0$ such that $x > N \Rightarrow$ $|\frac{2x-1}{x} - 2| < \epsilon$.
Proof: Given $\epsilon > 0$, choose $N = \underline{\frac{1}{\epsilon}}$.
For all $x > N$ we have:
$$|\frac{2x-1}{x} - 2| = |\frac{2x-1-2x}{x}| = |\frac{-1}{x}| = \frac{1}{|x|} < \frac{1}{N} = \frac{1}{1/\epsilon} = \epsilon;$$
therefore $|\frac{2x-1}{x} - 2| < \epsilon$ for all $x > N$. ■

Section 2.4

1. Hint: We don't yet have a rule for limits of compositions, or a rule for limits of functions that are more complicated than combinations of linear and integer power functions.

3. Hint: Look in the reading.

14. Hint: Does the limit of a function $f(x)$ as $x \to c$ have anything to do with the value $f(c)$?

17. Hint: Look in the reading.

Section 2.4 (continued)

26. The graph of $f(x) = 3$ is a horizontal line at $y = 3$. Given any ϵ-bar you draw around $y = 3$, *any* δ-bar around $x = 2$ will work (since the value of $f(x)$ is equal to 3, and thus within the ϵ-bar, no matter how far away you are from $x = 2$.

28. Hint: Sketch $f(x) = 1 - 3x$ and draw an ϵ-bar around the height $y = -11$. Look at the right triangle whose diagonal is the piece of the graph of $y = 1 - 3x$ that is trapped in this ϵ-bar. What does the slope of the line $y = 1 - 3x$ tell you about the lengths of the legs of this triangle? What does this mean in terms of δ and ϵ?

30. By the sum and constant multiple rules (applied in that order), we have:
$$\lim_{x\to 3}(2f(x) - 3g(x)) = \lim_{x\to 3} 2f(x) + \lim_{x\to 3}(-3g(x)) = 2\lim_{x\to 3} f(x) - 3\lim_{x\to 3} g(x).$$
$$= \lim_{x\to 3}(2f(x) - 3g(x)) = 2(5) - 3(-6) = 10 + 18 = 28.$$

31. It is not possible to calculate this limit at this point. First of all, we don't know what to do with limits of compositions (yet). Second of all, even if we *did* know how to take the limit of a composition, there is not enough information given in the problem.

32. It is not possible to calculate this limit with the information given. The values that $f(x)$ approaches as $x \to 3$ and as $x \to 4$ do not tell you anything about the value that $f(x)$ approaches as $x \to 7$!

34. The answer is $-\frac{1}{3}$. Show where you use each limit rule.

35. Part (a): Using the constant multiple and sum rules:
$$\lim_{x\to 1} 15(3 - 2x) = 15\lim_{x\to 1}(3 - 2x) = 15(\lim_{x\to 1}(3) + \lim_{x\to 1}(-2x))$$
$$= 15(\lim_{x\to 1}(3) - 2\lim_{x\to 1}(x)) = 15(3 - 2(1)) = 15.$$
The rules we used in each step (in order) were: the constant multiple rule, the sum rule, the constant multiple rule, and (together in the last step) the rules for the limit of a constant and the limit of the identity function.
Part (b): Now we aren't allowed to use the constant multiple or sum rules, so we'll do some algebra first and then apply the rule for the limit of a linear function:
$$\lim_{x\to 1} 15(3 - 2x) = \lim_{x\to 1}(45 - 30x) = 45 - 30(1) = 15.$$
The only limit rule we used was the rule for the limit of a linear function (at the second equals sign).

41. Writing out each step individually, we have:
$$\lim_{x\to -3} \frac{2}{3x+1} = \frac{\lim_{x\to -3} 1}{\lim_{x\to -3}(3x+1)} \quad \text{(quotient rule for limits)}$$
$$= \frac{1}{\lim_{x\to -3}(3x+1)} \quad \text{(limit of a constant)}$$
$$= \frac{1}{\lim_{x\to -3} 3x + \lim_{x\to -3} 1} \quad \text{(sum rule for limits)}$$
$$= \frac{1}{3\lim_{x\to -3} x + \lim_{x\to -3} 1} \quad \text{(constant multiple rule for limits)}$$
$$= \frac{1}{3(-3)+1} = -\tfrac{1}{8}. \quad \text{(limits of identity and constant functions)}$$

42. 6.

45. 4.

47. **Proof:** For all real numbers m, b, and c,
$$\lim_{x\to c}(mx + b) = \lim_{x\to c} mx + \lim_{x\to c} b \quad \text{(sum rule for limits)}$$
$$= m\lim_{x\to c} x + \lim_{x\to c} b \quad \text{(constant multiple rule for limits)}$$
$$= m(c) + b. \quad \text{(limits of constant and identity functions)} \blacksquare$$