Homework for Week 6 **Math 231 Fall 2001**

This homework will not be collected; it is your responsibility to do as many problems as necessary to understand the material. We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 2.5 1, 2, 3, 6, 8, 9, 11, 13, 14, 16, 17, 18, 20, 21, 22, 29, 30, 31, 32, 33, 34, 36, 37, 39, 41, 42, 43.

Section 2.6 1, 3, 5, 8, 9, 12, 14, 16, 17, 19, 20, 22, 24, 26, 28, 29, 31, 36, 37, 38, 40, 42, 45, 46, 48, 51, 54, 55, 56, 57, 60, 62, 64, 65, 66, 67.

Section 2.7 1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 18, 19, 20, 21, 27, 28, 33, 34, 37, 38, 39, 44, 48, 49, 52, 54, 56, 58, 59, 60, 62, 63.

Section 3.1 1, 2, 3, 4, 6, 8, 9, 11, 12, 15, 16, 17, 18, 19, 20, 22, 28, 34, 37, 39, 43, 45, 50, 51, 55, 57, 59, 60, 64, 66, 69, 72, 74, 78, 80, 81, 83, 86, 87, 91, 93, 94, 95, 96.

Selected Hints and Answers

Caution: The hints and answers below are not full solutions. Many of them would not be considered complete on a quiz or test.

Section 2.5

1. False; a counterexample would be a function whose limit at c is not equal to its value.
2. (D).
3. $f(x) = \sqrt{x+1}$ fails condition (a); $g(x) = \frac{1}{x-1}$ satisfies condition (a), but fails condition (b) at $x = 1$, since $g(1)$ is not a real number.
6. $(3x+1)(3x+1)(3x+1)$ (you could multiply this out, but it is not necessary).
8. Part (c): One example is $f(x) = 2x+1$, $g(x) = x^2+1$, and $c = 2$. A simpler example is $f(x) = x$, $g(x) = x$, and $x = 1$.
14. Writing out each step individually:

$$\begin{aligned}\lim_{x\to3}(3x + x^2(2x+1) &= \lim_{x\to3}(3x) + \lim_{x\to3}(x^2(2x+1)) && \text{(sum rule for limits)}\\ &= \lim_{x\to3}(3x) + \lim_{x\to3}(x^2)\cdot\lim_{x\to3}(2x+1) && \text{(product rule for limits)}\\ &= 3(3) + (3^2)(2(3)+1) && \text{(limit of linear and power functions)}\\ &= 9 + 9(7) = 72.\end{aligned}$$

Using Algorithm 1: $f(x) = 3x + x^2(2x+1)$ consists of linear and positive integer power functions that are added and multiplied together, so we can try to calculate the limit by evaluating $f(3)$. Since $f(3)$ turns out to be a real number, we have:

$$\lim_{x\to3}(3x + x^2(2x+1)) = 3(3) + 3^2(2(3)+1) = 9 + 9(7) = 72.$$

18. $\lim_{x\to0}\frac{x^2-1}{x-1} = \frac{0^2-1}{0-1} = \frac{-1}{-1} = 1$ (by Algorithm 1)
30. $\lim_{x\to2^-}\frac{x+1}{(x-2)^2}$, $\lim_{x\to2^+}\frac{x+1}{(x-2)^2}$, and $\lim_{x\to2}\frac{x+1}{(x-2)^2}$ are all $-\infty$ (by Theorem 2).
32. $\lim_{x\to0}\frac{x}{x^2-x} = \lim_{x\to0}\frac{1}{x-1} = -1$ (by Theorem 3 and Algorithm 1).
36. $\lim_{x\to-1^-} f(x) = -1$ and $\lim_{x\to-1^+} f(x) = 1$, so $\lim_{x\to1} f(x)$ does not exist.
41. Hint: Let $f(x) = mx + b$. Use the rule for taking the limit of a linear function, and then use the fact that $mc + b = f(c)$.

Section 2.5 (continued)

42. **Proof:** Suppose $f(x) = (mx+b)(kx+a)$ is a product of linear functions. Then for any real number c:

$$\begin{aligned}
\lim_{x\to c} f(x) &= \lim_{x\to c}(mx+b)(kx+a) && \text{(definition of } f(x)\text{)}\\
&= \lim_{x\to c}(mx+b)\cdot \lim_{x\to c}(kx+a) && \text{(product rule for limits)}\\
&= (mc+b)(kc+a) && \text{(limit of a linear function)}\\
&= f(c). && \text{(definition of } f(x)\text{)} \blacksquare
\end{aligned}$$

Section 2.6

3. f has a jump discontinuity at $x = 2$, and is right (but not left) continuous at $x = 2$.
5. f has a removable discontinuity at $x = 2$, and is neither right nor left continuous at $x = 2$.
12. It is not possible for a function to have the properties listed here; if f is both left and right continuous at $x = 1$ then f is necessarily continuous at $x = 1$.
17. This *is* possible; for example, consider the function $f(x) = \frac{1}{x-3}$. This function is continuous at each point in its domain, but is not continuous at the non-domain point $x = 3$.
19. Define $f(1)$ to be zero.
20. TYPO: Obviously that second inequality should be a $>$.
    Define $f(1)$ to be 2.
22. This statement is equivalent to the limit statement $\lim_{x\to c^+} f(x) = f(c)$, which by the formal definition of limit says that:

    For all $\epsilon > 0$, there exists a $\delta > 0$ such that $x \in (c, c+\delta) \;\Rightarrow\; |f(x) - f(c)| < \epsilon$.
24. 2.
28. No. Try drawing a function whose graph has a jump discontinuity at $x = 2$ with a "closed dot" on the left piece of the graph, and an "open dot" on the right piece of the graph.
29. False; find a counterexample.
31. It is equal to $f(1)$.
36. Discontinuous at $x = -1$ (removable) and at $x = 2$ (jump). Neither right nor left continuous at $x = -1$. Right but not left continuous at $x = 2$. Continuous on the intervals $(-\infty, -1)$, $(1, 2)$, and $[2, \infty)$.

' 37. Not continuous.

38. Amazingly, this function *is* continuous at $x = 0$; why?
45. $\lim_{x\to 1^-} f(x) = 2$ and $\lim_{x\to 1^+} f(x) = 4$, so $\lim_{x\to 1} f(x)$ does not exist. Since $f(1) = 2$, the function f is not continuous at $x = 1$. More precisely, f is left but not right continuous at $x = 1$.
48. TYPO: Again, the second inequality should be $\geq$. Argh!
    This function is is continuous at $x = 2$ (*i.e.* both left and right continuous).
51. $f(x)$ is a quotient of two continuous functions: its numerator is continuous because it is a power function with positive integer power; its denominator is continuous because it is a (constant multiple of) a linear function. Since the quotient of continuous functions is continuous, $f(x)$ is continuous.
54. $\lim_{x\to 0} |3x^2 - 5| = \left|\lim_{x\to 0}(3x^2 - 5)\right| = |3(0)^2 - 5| = |-5| = 5.$
    The first equality above uses Theorem 7.
56. $f(x)$ is *not* continuous at $x = 2$, since its limit does not exist (while $f(2) = 2 - 2 = 0$). The limit as $x \to 2$ does not exist because $f(x)$ approaches 0 when we consider rational values of x, but approaches 4 when we consider irrational values of x.
57. $f(x)$ is continuous at $x = 1$; as $x \to 1$ the function $f(x)$ approaches 1 whether we consider rational or irrational values, so $\lim_{x\to 1} f(x) = 1$. Since $x = 1$ is rational, $f(1) = 2 - 1 = 1$ and thus the function is continuous at $x = 1$.

Section 2.6 (continued)

60. Hint: Use the delta-epsilon definition of limit with $L = f(c)$.

61. **Proof:** Suppose $f(x) = mx + b$ is a linear function. Then for any value c in the domain $(-\infty, \infty)$ of $f(x)$, we have:
$$\lim_{x \to c} f(x) = \lim_{x \to c} mx + b = mc + b = f(c).$$
(The first equality is by the definition of f, the second equality is the rule for limits of linear functions, and the third equality is by the definition of f.) Since $\lim_{x \to c} f(x) = f(c)$ for all c in the domain of f, the function f is continuous. ■

64. Hint: This proof should look a lot like the one in Problem 61.

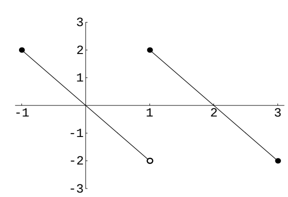
65. Hint: Do a delta-espilon proof that if $f(x) = 3x - 5$, then $\lim_{x \to 2} f(x) = f(2)$.

66. Since none of our limit rules apply to the absolute value function, the only way we can show $f(x) = |x|$ to be continuous is to do a delta-epsilon argument to show that $\lim_{x \to c} |x| = |c|$ for all real numbers c. We must show that:
$$\text{For all } \epsilon > 0, \text{ there exists a } \delta > 0 \text{ such that } 0 < |x - c| < \delta \ \Rightarrow\ ||x| - |c|| < \epsilon.$$
Proof: Given $\epsilon > 0$, choose $\delta = \underline{\epsilon}$.
If $0 < |x - c| < \delta$, then: $||x| - |c|| < |x - c|$ (by the inequality given in the hint)
$< \delta$ (by hypothesis)
$= \epsilon$ (by our choice of δ)
Therefore $||x| - |c|| < \epsilon$, which is what we wanted to show. ■

67. Hint: You must show that for all $c \in \text{Domain}(f) \cap \text{Domain}(g)$, we have $\lim_{x \to c}(f + g)(x) = (f + g)(c)$. Use the sum rule for limits.

Section 2.7

3. Part (a): See Theorem 2. The hypothesis is that f is continuous on a closed interval $[a, b]$. The conclusion is that for any number K between $f(a)$ and $f(b)$, there is at least one $c \in (a, b)$ such that $f(c) = K$.
Part (b): It says that a continuous function on a closed interval $[a, b]$ takes on all *intermediate values* between $f(a)$ and $f(b)$.
Part (c): The converse would say that: IF for any number K between $f(a)$ and $f(b)$, there is at least one $c \in (a, b)$ such that $f(c) = K$, THEN f must be continuous on the closed interval $[a, b]$. This converse is *not* true; a function might take on all the intermediate values between $f(a)$ and $f(b)$ and yet not be continuous.
Part (d): The contrapositive would say that: IF there is a number K between $f(a)$ and $f(b)$ for which there is *no* $c \in (a, b)$ with $f(c) = K$, THEN f is not continuous on the closed interval $[a, b]$. As always, the contrapositive is logically equivalent to the original theorem, and thus is true. It basically says that if a function skips over some value between $f(a)$ and $f(b)$, then that function could not have been continuous on $[a, b]$.

4. Part (c): Figure 5 in the reading is one example.
Part (d): One possible example, on the interval $[-1, 3]$, is:


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The existence of this example tells us that the converse of the Intermediate Value Theorm must be false (this function takes on all intermediate values betweewn $f(-1) = 2$ and $f(3) = -2$, but it is not continuous).

9. False.

11. True.

Section 2.6 (continued)

12. False.
13. True.
15. True.
16. False. (The function may have another zero somewhere in the interval $(0, 6)$; one counterexample is the function $f(x) = x(x-3)(x-6)$.)
19. This function is continuous on $[-2, 2]$ (since it is a polynomial), so the Extreme Value Theorem applies. On the interval $[-2, 2]$ this function has a maximum value of 2 at $x = -2$ and at $x = 2$, and a minimum value of approximately 4.25 occurring both at $x \approx -1.22$ and at $x \approx 1.22$.
27. This function is continuous on $[-2, 4]$ (since it is a polynomial), so the Intermediate Value Theorem applies. Moreover, $K = -4$ is between $f(-2) = -22$ and $f(4) = 14$; thus there is some $c \in (-2, 4)$ for which $f(c) = 4$. Using a graphing calculator we can approximate that $c \approx 3.5$ is such a value.
33. This function is continuous, since it is a polynomial. By the Intermediate Value Theorem, to show that there is some c for which $f(c) = -15$ we just need to find one value at which the height of the function is less than -15, and one value at which the height of the function is greater than -15. With a little trial and error we find that $f(-3) = (-3)^3 + 2 = -25$ and $f(0) = 2$; therefore there is some $c \in (-3, 0)$ for which $c = -15$. With a graphing calculator we can approximate that $c \approx -2.57$ is such a value.
37. The special case of the Intermediate Value Theorem applies because f is continuous on $[1, 2]$ (since it is a polynomial), and $f(1) = -2 < 0$ while $f(2) = 4 > 0$. Thus f has a root somewhere in $(1, 2)$. (With a graphing calculator we can see that one such root is approximately $x \approx 1.56$.
44. Hint: Don't forget to factor all the way first!
48. f is continuous and its only zeros are at $x = -2$ and $x = -\frac{1}{2}$. Checking the sign between each zero we have, for example, $f(-3) = 5 > 0$, $f(-1) = -1 < 0$, and $f(0) = 2 > 0$. Therefore f is positive on $(-\infty, -2)$ and on $(-\frac{1}{2}, \infty)$, and negative on $(-2, -\frac{1}{2})$.
56. Hint: f is *not* discontinuous at $x = 1$; why?
58. f is discontinuous at $x = 2$ and has only one root, at $x = 0$ (why?). Testing the sign of f between these values we get, for example, $f(-1) = -1 < 0$, $f(1) = 1 > 0$, and $f(3) = 3 > 0$. Therefore f is positive on $(0, 2)$ and $(2, \infty)$, and in this case is also positive at $x = 2$ since $f(2) = 8$; hence f is positive on $(0, \infty)$. On the other hand, f is negative on $(-\infty, 0)$. Sketch a graph of this function to verify these answers!
59. The length $H(t)$ of Alina's hair is a continuous function (her hair can't suddenly get longer or shorter without going through a continuous change), so the EVT and IVT will apply to this function. We are given that $H(0) = 2$ and $H(6) = 42$. The Extreme Value Theorem says that Alina's hair had a maximum and a minimum length sometime in the last six years. The Intermediate Value Theorem says that Alina's hair was every length between 2 and 42 inches at some point during the last six years.

Section 3.1

8. Part (a): c is the x-coordinate of the first dot.
   Part (b): h is the horizontal distance between the first and second dots.
   Part (c): $c + h$ is the x-coordinate of the second dot.
   Part (d): $f(c)$ is the y-coordinate of the first dot.
   Part (e): $f(c + h)$ is the y-coordintae of the second dot.
   Part (f): $f(c + h) - f(c)$ is the vertical distance between the first and second dots.
   Part (g): $\frac{f(c+h)-f(c)}{h}$ is the slope of the line connecting the two dots ("rise over run").
   Part (h): $f'(c)$ is the slope of the line tangent to the graph at the first dot.

Section 3.1 (continued)

11. Hint: Now the "x" is representing the "$c+h$;" one consequence of this is that $x-c$ is playing the role of the small distance h.

16. Hint: Define $x := c+h$; then $h = x-c$. As $h \to 0$, what happens to x?

17. In the alternate notation, the left derivative is $f'_-(c) := \lim_{x\to c^-} \dfrac{f(x)-f(c)}{x-c}$.

18. ... *exist* and are *equal*.

19. Notice that this time the variable in the limit is h (as opposed to the usual "x")! In terms of deltas and espilons, the definition of derivative says that:

    For all $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |h-0| < \delta \Rightarrow |\frac{f(c+h)-f(c)}{h} - f'(c)| < \epsilon$.

20. This time the variable in the limit is the familiar "x;" in terms of the delta-epsilon definition of limit, the alternate definition of derivative says that:

    For all $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |x-c| < \delta \Rightarrow |\frac{f(x)-f(c)}{x-c} - f'(c)| < \epsilon$.

22. $\dfrac{f(x)-f(c)}{x-c} = \dfrac{f(1.9)-f(2)}{1.9-2} = -11.41.$

28. $\dfrac{f(c+h)-f(c)}{h} = \dfrac{f(2-0.1)-f(2)}{-0.1} = -11.41.$

37. I'll compute the slopes of *four* secant lines from $(c, f(c))$ to $(c+h, f(c+h))$:

    When $h = 0.1$, the slope is ≈ -0.13158.
    When $h = 0.001$, the slope is ≈ -0.12506.
    When $h = -0.001$, the slope is ≈ -0.12494.
    When $h = -0.1$, the slope is ≈ -0.11905.

    It appears that as $h \to 0$ we might be approaching a slope of about -0.125 (think about what number would fit in between the second and third calculations). We look at a number of values of h approaching 0 (from the left and right) so we can be reassured that the slope actually seems to be *approaching* some number as $h \to 0$.

43. By the definition of derivative, the *exact* value $f'(-2)$ of the slope of the tangent line to $f(x) = \frac{1}{2x}$ at $x = -2$ is:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(-2+h)-f(-2)}{h} &= \lim_{h\to 0} \frac{\frac{1}{2(-2+h)} - \frac{1}{2(-2)}}{h} && \text{(definitions of } c \text{ and } f\text{)} \\
&= \lim_{h\to 0} \frac{\frac{1}{-4+2h} + \frac{1}{4}}{h} && \text{(simplify)} \\
&= \lim_{h\to 0} \frac{\frac{4+(-4+2h)}{4(-4+2h)}}{h} && \text{(combine fractions)} \\
&= \lim_{h\to 0} \frac{2h}{4h(-4+2h)} && \text{(algebra)} \\
&= \lim_{h\to 0} \frac{1}{2(-4+2h)} && \text{(cancel)} \\
&= \frac{1}{2(-4+2(0))} = \frac{1}{2(-4)} = -\tfrac{1}{8}. && \text{(limit rules)}
\end{aligned}$$

    Note that this is the *exact* answer to the approximation you did in Problem 37.

51. $f'(-3) = \lim_{h\to 0} \dfrac{f(-3+h)-f(-3)}{h} = \lim_{h\to 0} \dfrac{(-3+h)^2-(-3)^2}{h} = \lim_{h\to 0} \dfrac{(9-6h+h^2)-9}{h}$
    $= \lim_{h\to 0} \dfrac{-6h+h^2}{h} = \lim_{h\to 0} \dfrac{h(-6+h)}{h} = \lim_{h\to 0} (-6+h) = -6+0 = -6.$

57. -1.

60. $f'(-3) = \lim_{x\to -3} \dfrac{f(x)-f(-3)}{x-(-3)} = \lim_{x\to -3} \dfrac{x^2-(-3)^2}{x+3} = \lim_{x\to -3} \dfrac{x^2-9}{x+3} = \lim_{x\to -3} \dfrac{(x-3)(x+3)}{x+3}$
    $= \lim_{x\to -3} (x-3) = -3-3 = -6.$

    Note that this is the same derivative we found in Problem 51, but since we were using the alternate definition here, the algebra was much different!

Section 3.1 (continued)

69. The line we are looking for passes through the point $(-3, f(-3)) = (-3, (-3)^2) = (-3, 9)$ and has slope $f'(-3) = -6$ (by Problem 51 or 60). Using the point-slope form of a line we can easily find the equation of this line: $y - 9 = -6(x - (-3))$. In slope-intercept form this equation becomes: $y = -6x - 9$. Use your calculator to graph of $f(x) = x^2$ and this line to verify that this is in fact the tangent line to the graph of f at $x = -3$.

74. Here we are using the original definition of derivative, with $f(x) = 3x^2 + 1$ and $c = -1$ (note that $f(-1) = 3(-1)^2 + 1 = 4$).

75. Here we are using the alternate definition of derivative, with $f(x) = x^4$ and $c = 3$ (note that $f(3) = 3^4 = 81$).

80. $$f'_-(2) = \lim_{h\to 0^-} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^-} \frac{((2+h)+4) - 3(2)}{h} = \lim_{h\to 0^-} \frac{2+h+4-6}{h}$$
$$= \lim_{h\to 0^-} \frac{h}{h} = \lim_{h\to 0^-} 1 = 1.$$
$$f'_+(2) = \lim_{h\to 0^+} \frac{f(2+h) - f(2)}{h} = \lim_{h\to 0^+} \frac{3(2+h) - 3(2)}{h} = \lim_{h\to 0^+} \frac{6+3h-6}{h}$$
$$= \lim_{h\to 0^+} \frac{3h}{h} = \lim_{h\to 0^+} 3 = 3.$$
Since $f'_-(2)$ and $f'_+(2)$ are not equal, $f'(2)$ does not exist. (Verify the calculations above with a graph of $f(x)$!)

91. The best possible left-sided estimate for $f'(3)$ is $\frac{f(3)-f(2)}{3-2} = \frac{0-2}{3-2} = -2$.
The best possible right-sided estimate for $f'(3)$ is $\frac{f(4)-f(3)}{4-3} = \frac{-4-0}{4-3} = -4$.
The best possible two-sided estimate for $f'(3)$ is $\frac{f(4)-f(2)}{4-2} = \frac{-4-2}{4-2} = -3$.