Homework for Week 7 **Math 231 Fall 2001**

We recommend that you read each section before attempting any exercises. This homework will not be collected, but next week's quiz will be a subset of the problems below.

Special Note: There are less problems assigned this week than there have been in previous weeks. If you feel you need more practice to understand the concepts in these sections, then do more exercises.

Section 3.2 2, 6, 7, 8, 10, 16, 19, 26, 29, 30, 31, 32, 33.

Section 3.3 1, 5, 7, 10, 11, 12, 14, 16, 21, 27, 28, 34, 35, 36.

Section 3.4 1, 7, 8, 12, 13, 19, 23, 26, 29, 32, 35, 38, 40, 44, 46, 48, 49, 51, 52.

Selected Hints and Answers

Caution: The hints and answers below are not full solutions. Many of them would not be considered complete on a quiz or test.

Section 3.2

6. $(c) < (d) < (a) < (b)$
7. Greatest IROC at approximately $x = 0.5$; least IROC at approximately $x = -1.5$ or $x = 2$ (all the way to the left or right of the given graph); IROC is zero at approximately $x = -0.4$ and $x = 1.3$.
8. Positive IROC on $(0, \infty)$; negative IROC on $(-\infty, 0)$; IROC of zero at $x = 0$.
10. Your graph should have a horizontal tangent line at $x = 2$. The slope of the line connecting the points $(1, f(1))$ and $(2, f(2))$ on the graph should have a slope of 2 (think of "rise over run"). The slope from $(2, f(2))$ to $(3, f(3))$ should be 1.
16. Approximately -1.111.
19. AROC on $[1, 1.1]$ is -2.1, AROC on $[1, 1.01]$ is -2.01, AROC on $[1, 1.001]$ is -2.001; these values seem to approach 2. From the left, AROC on $[0.9, 1]$ is -1.9, AROC on $[0.99, 1]$ is -1.99, AROC on $[0.999, 1]$ is -1.999; these values also seem to approach 2. From this we can aproximate that the IROC at $x = 1$ might be 2.
26. Using the definition of derivative to find $f'(-2)$ you should get:
$$f'(-2) = \lim_{h\to 0} \frac{f(-2+h) - f(-2)}{h} = \lim_{h\to 0} \frac{\frac{1}{(-2+h)+1} - (-1)}{h} = \ldots \text{(work)} \ldots = -1.$$
29. 5 miles per hour. (Use $d = rt$ formula. Watch out for your units; the rate is given in miles per hour but the time is given in *minutes*.)
30. Part (a): For example, the AROC of the number of drive-in movie theatres from 1967 to 1972 is 2.4 drive-ins per year.
31. Part (a): Her AROC was zero on $[0, 30]$; this means that after 30 minutes she was exactly as far away from the oak tree as she was at time $t = 0$.
    Part (b): The second ten minutes.
    Part (c): One approximation might be $\frac{140-280}{10-5} = -28$ feet per minute. The fact that the sign is negative indicates that she is moving *towards* the oak tree.
32. Part (a): $s(t) = 0$ at approximately $t = 1.62$ seconds, so the average velocity of the orange is $\frac{s(1.62)-s(0)}{1.62-0} \approx -25.93$ feet per second.
    Part (b): AROC of $s(t)$ on $[1.6, 1.62]$ is ≈ -51.52; AROC of $s(t)$ on $[1.61, 1.62]$ is ≈ -51.68; AROC of $s(t)$ on $[1.619, 1.62]$ is ≈ -51.824. Thus IROC of $s(t)$ at $x = 1.62$ (the time the orange hits the ground) might be approximately -51.825 feet per second.

Section 3.2 (continued)

32. (continued)

Part (c): Using the definition of derivative,

$$v(1.62) = s'(1.62) = \lim_{h\to 0} \frac{s(1.62+h) - s(1.62)}{h} = \ldots(\text{work})\ldots = -51.84\ \tfrac{\text{feet}}{\text{sec}}.$$

Part (d): Using the definition of derivative,

$$a(1.62) = v'(1.62) = \lim_{h\to 0} \frac{v(1.62+h) - v(1.62)}{h} = \ldots(\text{work})\ldots = -32\ \tfrac{\text{feet}}{\text{sec}^2}.$$

33. $h(12)$ is the average height of a 12-year-old, in feet. $h'(12)$ is measured in feet per year, and represents the instantaneous rate of change of the height of a 12-year-old person; that is, $h'(12)$ is the rate at which an average 12-year-old is growing (in feet per year).

Section 3.3

1. f is differentiable, and thus continuous, at $x = c$.
5. f is continuous, but not differentiable, at $x = 1$.
7. $\displaystyle\lim_{h\to 0} \frac{(4(2+h)^3 - 5(2+h) + 1) - 23}{h}$ exists.
10. False.
11. False.
12. True.
14. Not differentiable at $x = -1$. Left but not differentiable at $x = -1$. (Left derivative is some largish number like 20, while the slopes of the secant lines for the right derivative approach ∞; why?)
21. You could write f as a piecewise function from the start, and then look at the left and right derivatives of f at $x = -1$, or you could look at the limit:

$$\lim_{h\to 0} \frac{|(-1+h)^2 - (-1+h) - 2| - 0}{h} = \lim_{h\to 0} \frac{|h||h-3|}{h}$$

from the left and the right. The one-sided limits are:

$$\lim_{h\to 0^-} \frac{|h||h-3|}{h} = (-1)|0-3| = -3 \quad \text{and} \quad \lim_{h\to 0^+} \frac{|h||h-3|}{h} = (1)|0-3| = 3.$$

Thus f is left *and* right differentiable at $x = -1$, but *not* differentiable at $x = -1$.

27. Part (a): The left and right limits of $f(x)$ as $x \to 1$ are:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1 \quad \text{and} \quad \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (2x-1) = 1,$$

so f is continuous at $x = 1$.

Part (b): The left and right derivatives of f at $x = 1$ are:

$$f'_-(1) = \lim_{h\to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^-} \frac{(1+h)^2 - 1}{h} = 2,$$

$$f'_+(1) = \lim_{h\to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0^+} \frac{(2(1+h) - 1) - 1}{h} = 1.$$

Therefore f is *not* differentiable at $x = 1$.

28. Once you show that f is not continuous at $x = 1$ you also know that f must not be differentiable at $x = 1$.
34. Do this problem graphically: sketch a graph of $f(x)$ and consider the secant lines from $(1, f(1))$ to $(1+h, f(1+h))$ as $h \to 0$. Consider the case where $1+h$ is rational separately from the case where $1+h$ is irrational.
35. Part (a): $S(3) = 200 + 8(10)(3) = \$440$, $S(6) = 200 + 8(10)(6) = \$680$, and $S(8) = 200 + 8(10)(6) + 11.5(10)(2) = \$910$.

Part (b): $S'(3)$ is measured in dollars per week, and represents the rate of change of your savings account, in dollars per week, at the start of the third week. Since you are being paid $8(10) = \$80$ per week at that time, $S'(3) = 80$ dollars per week. Similarly, $S'(8) = 11.5(10) = \$115$ dollars per week. However, we cannot compute $S'(6)$ because the rate at which you are being paid changes at that point.

Section 3.3 (continued)

35. (continued)

Part (c): $S(t) = \begin{cases} 200 + 8(10)t, & t \leq 6 \\ 680 + 11.5(10)(t-6), & t > 6. \end{cases}$

36. Use the fact that if the left and right derivatives $f'_-(c)$ and $f'_+(c)$ exist and are equal, then $f'(c)$ exists (and is equal to the same thing; why?), and thus f is differentiable at $x = c$.

Section 3.4

1. The height of the graph $y = f'(x)$ at $x = c$ is the same as the slope of the tangent line to the graph of $y = f(x)$ at $x = c$.
7. Your graph of $y = f'(x)$ should have zeros at $x = -1.5$, $x = 0$, and $x = 1.5$, should be positive on $(-\infty, -1.5)$ and $(0, 1.5)$, and should be negative on $(-1.5, 0)$ and $(1.5, \infty)$.
8. At $x = 1$ your graph should have a height of 2 and a horizontal tangent line. At $x = 3$ your graph should be drawn so that its tangent line has a slope of 2.
12. Figure 17 is the graph of f, Figure 18 is the graph of f', and Figure 16 is the graph of f''. Make sure you can articulate why this is so using the relationship between the slope of the graph of a function and the height of the graph of its derivative.
13. Your graph should have horizontal tangent lines at $x = 2$ and at $x = 0$. Moreover, the tangent lines to the graph of your function should have a negative slope everywhere on $(-\infty, -2)$, a positive slope everywhere on $(-2, 0)$, and a negative slope everywhere on $(0, \infty)$.
19. $\left.\dfrac{d^4}{dx^4}\right|_{-4} (3x^2 - 1)$.
23. The answer is $\frac{-1}{x^2}$, using the reciprocal rule, the constant multiple rule (with x as the "constant"), the sum rule, and the rule for limits of constant and identity functions (with h as the "identity function").
26. $f'(x) = \lim_{h \to 0} \dfrac{((x+h)^2 - 4) - (x^2 - 4)}{h} = \ldots \text{(work)} \ldots = 2x$.
29. $f'(x) = \lim_{h \to 0} \dfrac{(1 + (x+h) + (x+h)^2) - (1 + x + x^2)}{h} = \ldots \text{(work)} \ldots = 1 + 2x$.
32. $f'(x) = \lim_{h \to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h} = \ldots \text{(work)} \ldots = \dfrac{-1}{x^2}$.
35. The tangent line to $f(x)$ at $x = -2$ has slope $f'(-2) = 1 + 2(-2) = -3$ and passes through $(-2, f(-2)) = (-2, 3)$. Therefore its equation is $y - 3 = -3(x + 2)$, or in slope-intercept form, $y = -3x - 3$.
38. Clearly the graph of f' should have a root at $x = 0$, and be positive on $(-\infty, 0)$ and negative on $(0, \infty)$. There is one more thing to notice here, however; the graph of f gets very "flat" at its ends, so its tangent lines have very shallow slopes as $x \to \infty$ and as $x \to -\infty$. Therefore the graph of the derivative should be approaching zero as $x \to \infty$ and as $x \to -\infty$ (why?). Thus your graph of f' should also have a two-sided horizontal asymptote at $y = 0$.
40. Here $f(x) = 2x^3$. Calculating the limit gives us $f'(x) = 6x^2$.
44. Using the definition of derivative you can show that $f'(x) = 6x$. Then using the definition of derivative to differentiate *that* you can show that $f''(x) = 6$. Finally, using the definition of derivative to differentiate the second derivative, you can show that $f'''(x) = 0$.
46. Since $f'''(x) = 0$ for all x, we have $f'''(-2) = 0$.
48. TYPO: All the x's should be t's.

Part (a): $s(t) = 0$ at $t = 2$ and $t = -6$. Thus it will take 2 hours to reach the Donut Hole.

Part (b): $v(t) = s'(t) = \lim_{h \to 0} \dfrac{(-10(t+h)^2 - 40(t+h) + 120) - (-10t^2 - 40t + 120)}{h}$

$= \ldots \text{(work)} \ldots = -20t - 40.$

48. (continued)

Part (c): $a(t) = v'(t) = \lim_{h \to 0} \dfrac{(-20(t+h) - 40) - (-20t - 40)}{h} = \ldots\text{(work)}\ldots = -20.$
Since the negative direction is *towards* the Donut Hole (because $s(t)$ measures the distance *away* from the Donut Hole), and $a(t)$ is negative, you are speeding up as you approach the Donut Hole (*i.e.* accelerating), at a rate of 20 miles per hour per hour.

49. Part (a): Your graph should look like an upside-down parabola with roots at $x = 0$ and at $x = 2$. The physical interpretation is that your velocity is always positive, and that you start out with a zero velocity, speed up, and then eventually slow back down to a zero velocity.
Part (b): Your graph should have a root at about $x = 1$, and be positive on $[0, 1)$ and negative on $(1, 2]$ (for example, a line with negative slope passing through $(1, 0)$). This represents the fact that you were speeding up, and then slowing down, during your trip.

51. This is a computational proof; if $f(x) = mx + b$ then:

$$f'(x) = \lim_{h \to 0} \frac{(m(x+h) + b) - (mx + b)}{h} = \ldots\text{(work)}\ldots = m.$$

52. Hint: What is the equation of the line with slope $f'(c) = m$ that passes through $(c, f(c)) = (c, mc + b)$?