Homework for Week 10 **Math 232 Spring 2002**

This homework will not be collected. It is your responsibility to do as many problems as necessary to understand the material (this includes doing extra problems if you need more practice). We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 12.1 1, 6, 7, 10, 15, 18, 20, 21, 23, 24, 25, 28, 33, 35, 37, 40, 43, 45, 48, 50, 53, 57, 59.

Section 12.2 1, 3, 5, 9, 10, 11, 12, 13, 15, 17, 18, 24, 28, 29, 30, 31, 35, 38, 39, 40, 42, 43*.

Section 12.3 2, 3, 4, 7, 8, 9, 11, 16, 17, 19, 21, 25, 26, 29, 31, 32, 34, 37, 41, 42, 43.

Selected Hints and Answers

Caution: The hints and answers below are not necessarily full solutions. Many of them would not be considered complete on a quiz or test. **Answers are not provided for problems whose answers can be found in the reading or problems whose answers are easy to check using a calculator.**

Section 12.1

10. One possible answer is $a_k = k^2$, $m = 3$, $n = 7$; can you think of another?

15. **(a)** $\sum_{k=2}^{6} \frac{2}{(k-1)^2}$, **(b)** $\sum_{k=1}^{5} \frac{2}{k^2}$, **(c)** $\sum_{k=0}^{4} \frac{2}{(k+1)^2}$.

18. False. (Try $n = 3$.)

20. False. (How could you change it to be true?)

21. False. (Try it!)

25. The first rectangle has a height of $f(1) = 1$ unit and a width of $\frac{1}{2}$ unit; therefore the area of the first rectangle is $(1)(\frac{1}{2}) = 0.5$. The height of the second rectangle is $f(1.5) \approx 1.25$ (use the formula $f(x) = x^2 - 2x + 2$); this rectangle also has a width of $\frac{1}{2}$ units, so its area is $(1.25)(\frac{1}{2}) = 0.75$. By the same method the third rectangle has an area of 1 square unit (its height is $f(2) = 2$ units), and the fourth rectangle has an area of 1.625 units (its height is $f(2.5) = 3.25$ units). Therefore the sum of the areas of all the rectangles is $0.5 + 0.75 + 1 + 1.625 = 3.875$ square units.

28. 271.

33. 1.8558.

35. $\sum_{k=1}^{8} 3$ ($m = 1$, $n = 8$, $a_k = 3$).

37. $\sum_{k=1}^{5} \frac{k+2}{k^3}$ and $\sum_{k=3}^{7} \frac{k}{(k-2)^3}$ are the most obvious ways.

40. $2\left(a_0 + a_1 + a_2 + \sum_{k=3}^{100} a_k\right) - \left(\sum_{k=3}^{100} a_k + a_{101}\right) = 2a_0 + 2a_1 + 2a_2 - a_{101} + \sum_{k=3}^{100} a_k$.

43. 11.

45. 48.

Section 12.1 (continued)

48. $$\sum_{k=3}^{n}(k+1)^2 = \sum_{k=1}^{n}(k+1)^2 - (1+1)^2 - (2+1)^2 = \sum_{k=1}^{n}k^2 + 2\sum_{k=1}^{n}k + \sum_{k=1}^{n}1 - 13$$
$$= \frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2} + n - 13.$$
When $n = 100$ the sum is: $\dfrac{100(101)(201)}{6} + 2\dfrac{100(101)}{2} + 100 - 13 = 348,537.$
When $n = 500$ the sum is: $42,042,737$.
When $n = 1000$ the sum is: $334,835,487$.

50. $$\sum_{k=1}^{n}\frac{k^3-1}{n} = \frac{1}{n}\left(\sum_{k=1}^{n}k^3 - \sum_{k=1}^{n}1\right) = \frac{1}{n}\left(\frac{n^2(n+1)^2}{4} - n\right).$$

53. $$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k^2+k+1}{n^3} = \lim_{n\to\infty}\left(\frac{1}{n^3}\left(\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k + \sum_{k=1}^{n}1\right)\right)$$
$$= \lim_{n\to\infty}\left(\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n\right)\right)$$
$$= \lim_{n\to\infty}\frac{2n^3+6n^2+10n}{6n^3} = \frac{1}{3}.$$

57. $$\lim_{n\to\infty}\sum_{k=1}^{n}(1+\tfrac{k}{n})^2\cdot\tfrac{1}{n} = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{k=1}^{n}1 + \frac{2}{n^2}\sum_{k=1}^{n}k + \frac{1}{n^3}\sum_{k=1}^{n}k^2\right)$$
$$= \lim_{n\to\infty}\left(\frac{1}{n}\cdot n + \frac{2}{n^2}\cdot\frac{n(n+1)}{2} + \frac{1}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right)$$
$$= \lim_{n\to\infty}\left(1 + \frac{n^2+n}{n^2} + \frac{2n^3+3n^2+n}{6n^3}\right) = 1+1+\frac{2}{6} = \frac{7}{3}.$$

59. Mimic the proof of Theorem 1(b).

Section 12.2

1. True. (Why?)
3. False. (Can you draw a picture?)
5. True. (Why?)
9. **(a)** 7.85; **(b)** no; **(c)** Upper Sum is ≥ 8.2, Lower Sum is ≤ 7.5; **(d)** no; **(e)** no.
10. Hint: Compare the heights of the rectangles used with the Upper Sum to the heights of the rectangles used for any other Riemann sum; what does this imply about the sums of areas?
13. Right Hand Sum with $f(x) = x^2$, $N = 4$, $\Delta x = \frac{1}{2}$, and $x_k = 1 + \frac{k}{2}$; thus $a = 1$ and $b = 3$.
15. Trapezoid Sum with $f(x) = \sin x$, $N = 100$, $\Delta x = 0.05$, and $x_k = 0 + 0.05k$ (so $x_{k-1} = 0 + 0.05(k-1)$); thus $a = 0$ and $b = 0 + 0.05(100) = 5$.
17. Part of the sum would indicate that $\Delta x = \frac{1}{2}$, while another part would give us $\Delta x = 0.25$.
18. If $N = 100$, $\Delta x = 0.1$, and $a = 2$, then $b = 2 + 100(0.1) = 12$, so we can't have $[a, b] = [2, 5]$.
24. Try graphing the function and then using the "area" calculation from $x = 1$ to $x = 4$; you should get approximately 51.88.
28. Part (a): $f(0)(1) + f(1)(1) + f(2)(1) = 5$.
    Part (b): $f(0)(0.5) + f(0.5)(0.5) + f(1)(0.5) + f(1.5)(0.5) + f(2)(0.5) + f(2.5)(0.5) = 6.875$.
29. Part (a): $f(2)(0.25) + f(2.25)(0.25) + f(2.5)(0.25) + f(2.75)(0.25) \approx 1.16641$.
    Part (b): $f(2.25)(0.25) + f(2.5)(0.25) + f(2.75)(0.25) + f(3)(0.25) \approx 1.26997$.

Section 12.2 (continued)

30. Part (a): $f(1.25)(0.5) + f(1.75)(0.5) + f(2.25)(0.5) + f(2.75)(0.5)$
$+f(3.25)(0.5) + f(3.75)(0.5) \approx 51.3434.$
Part (b): $\frac{f(1)+f(1.5)}{2}(0.5) + \frac{f(1.5)+f(2)}{2}(0.5) + \frac{f(2)+f(2.5)}{2}(0.5) + \frac{f(2.5)+f(3)}{2}(0.5)$
$+\frac{f(3)+f(3.5)}{2}(0.5) + \frac{f(3.5)+f(4)}{2}(0.5) \approx 52.9562.$

31. Part (a): $f(2)(1) + f(2)(1) = 2.$
Part (b): $f(\frac{5}{3})(\frac{2}{3}) + f(2)(\frac{2}{3}) + f(\frac{7}{3})(\frac{2}{3} \approx 2.148.$
Part (c): $f(1.5)(0.5) + f(2)(0.5) + f(2)(0.5) + f(2.5)(0.5) = 2.25.$

35. $\sum_{k=0}^{2} k^2 \quad (\text{or } \sum_{k=1}^{3}(k-1)^2).$

38. $\sum_{k=1}^{3} \frac{\sin((k-1)\frac{\pi}{3}) + \sin(k \cdot \frac{\pi}{3})}{2} \left(\frac{\pi}{3}\right).$

39. $\sum_{k=0}^{99} \ln(2 + \frac{3}{100}k) \quad (\text{or } \sum_{k=1}^{100} \ln(2 + \frac{3}{100}(k-1))).$

40. $\sum_{k=1}^{20} \sqrt{(-1 + \frac{2k-1}{20})^2 - 1}\,(\frac{1}{10}) \quad (-1 + \frac{2k-1}{20}$ is $\frac{x_{k-1}+x_k}{2}$ simplified, where $x_k = -1 + \frac{k}{10}).$

42. Exact = 26 (triangle and rectangle); LHS = 24.5; RHS = 27.5; Midpoint = 26; Upper = 27.5; Lower = 24.5; Trapezoid = 26.

43. TYPO: The function should be $f(x) = \sqrt{1-x^2}$.
Exact = $\frac{\pi}{2}$ (half of circle); LHS ≈ 1.366; RHS ≈ 1.366; Midpoint ≈ 1.630; Upper ≈ 1.866; Lower ≈ 0.866; Trapezoid ≈ 1.366.

Section 12.3

7. Hint: Compare the height of a rectangle for the Right Hand Sum of a function $f(x)$ with the height of the corresponding rectangle in the Right Hand Sum for $kf(x)$.

8. False.

9. True.

10. False.

16. $\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$, but $\int_a^b f(x)\cdot g(x)\,dx \neq \int_a^b f(x)\,dx \cdot \int_a^b g(x)\,dx$.
Hint: For parts (b) and (c) pick really simple functions like lines so that you can calculate the definite integrals exactly.

18. Part (a): RHS $= \sum_{k=1}^{N} (\frac{3k}{N})^2\,(\frac{3}{N}) = \frac{27}{N^3}\left(\frac{N(N+1)(2N+1)}{6}\right).$
Part (b): LHS $= \sum_{k=1}^{N} (\frac{3}{N}(k-1))^2\,(\frac{3}{N}) = \frac{27}{N^3}\left(\frac{N(N+1)(2N+1)}{6} - 2\frac{N(N+1)}{2} + N\right).$
Part (c): For $N = 100$, RHS ≈ 9.13545 while LHS ≈ 8.86545; for $N = 1000$, we have RHS ≈ 9.0135 while LHS ≈ 8.9865.
Part (d): The expressions in parts (a) and (b) are both equal to 9 if we let $N \to \infty$.

25. $\sum_{k=1}^{N}(5 - (2 + \frac{3k}{N}))(\frac{3}{N}) = \frac{3}{N}\sum_{k=1}^{N} 3 - \frac{9}{N^2}\sum_{k=1}^{N} k = \frac{3}{N}(3N) - \frac{9}{N^2}\left(\frac{N(N+1)}{2}\right)$. When $N = 100$ the sum is ≈ 4.455; when $N = 1000$ the sum is ≈ 4.4955; the limit as $N \to \infty$ is $\frac{9}{2}$.

Section 12.3 (continued)

26. $\displaystyle\sum_{k=1}^{N} 2(\tfrac{k}{N})^2(\tfrac{1}{N}) = \frac{2}{N^3}\sum_{k=1}^{N} k^2 = \frac{2}{N^3}\left(\frac{N(N+1)(2N+1)}{6}\right)$. When $N = 100$ the sum is ≈ 0.6767; when $N = 1000$ the sum is ≈ 0.667667; the limit as $N \to \infty$ is $\frac{2}{3}$.

29. $\displaystyle\sum_{k=1}^{N}((2+\tfrac{k}{N})+1)^2(\tfrac{1}{N}) = \frac{1}{N}\sum_{k=1}^{N} 9 + \frac{6}{N^2}\sum_{k=1}^{N} k + \frac{1}{N^3}\sum_{k=1}^{N} k^2 = \frac{1}{N}(9N) + \frac{6}{N^2}\left(\frac{N(N+1)}{2}\right) + \frac{1}{N^3}\left(\frac{N(N+1)(2N+1)}{6}\right)$. When $N = 100$ the sum is ≈ 12.3684; when $N = 1000$ the sum is ≈ 12.3368; the limit as $N \to \infty$ is $9 + \frac{6}{2} + \frac{2}{6} = \frac{37}{3}$.

31. $9 - 4 = 5$.

32. $9 + 2(3) = 14$.

34. $2 \cdot 4 - 2 = 6$.

37. Not enough information.

41. $-(2 \cdot \frac{1}{2}(3^2 - 2^2) - 3(2)) = 1$.

42. Part (a): $\displaystyle\lim_{N\to\infty}\sum_{k=1}^{N}(3(1+\tfrac{2k}{N})+4)(\tfrac{2}{N}) = \lim_{N\to\infty}\left(\frac{2}{N}(7N) + \frac{12}{N^2}\cdot\frac{N(N+1)}{2}\right) = 14 + \frac{12}{2} = 20$.

    Part (b): The area is a rectangle with base of length 2 and height of length 7, with a triangle on top with base 2 and height 6; therefore the area is: $2(7) + \frac{1}{2}(2 \cdot 6) = 20$.

    Part (c): $\displaystyle\int_1^3 (3x+4)\,dx = 3\int_1^3 x\,dx + 4\int_1^3 1\,dx = 3 \cdot \tfrac{1}{2}(3^2 - 1^2) + 4 \cdot 1(3-1) = 20$.

43. See outline given in the reading; the proof is similar to the proof of Theorem 1(a).