Homework for Week 12 **Math 232 Spring 2002**

This homework will not be collected. It is your responsibility to do as many problems as necessary to understand the material (this includes doing extra problems if you need more practice). We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 13.2 1, 2, 3, 4, 5, 9, 10, 14, 15, 16, 17, 22, 23, 26, 27, 30, 32, 38, 39, 40.

Section 13.3 1, 2, 3, 4, 6, 8, 10, 11, 13, 14, 15, 16, 19, 23, 26, 30, 33, 34, 37, 39, 42, 44.

Section 14.1 4, 6, 10, 13, 15, 16, 19, 21, 22, 26, 29, 39, 40, 41, 48, 49, 54, 57, 65, 66, 69, 74, 77.

Selected Hints and Answers

Caution: The hints and answers below are not necessarily full solutions. Many of them would not be considered complete on a quiz or test. **Answers are not provided for problems whose answers can be found in the reading or problems whose answers are easy to check using a calculator.**

Section 13.2

3. Hint: Where do we use the fact that f is continuous in the proof of the Fundamental Theorem?
9. Equivalent to FTC: (a), (c), (d), (f), and (g).
10. Part (a): $f(1.1)(.1) + f(1.2)(.1) + \ldots + f(1.9)(.1) + f(2)(.1) = 7.455$ (Right Hand Sum).

    Part (b): $\int_1^2 3x^2 = 3\int_1^2 x^2\,dx = 3(\frac{1}{3})(2^3 - 1^3) = 7.$

    Part (c): $$\int_1^2 3x^2 = \lim_{N\to\infty} 3(1+\tfrac{k}{N})^2(\tfrac{1}{N}) = \lim_{N\to\infty} \tfrac{3}{N}\sum_{k=1}^{N}(1+\tfrac{2k}{N}+\tfrac{k^2}{N^2})$$
    $$= \lim_{N\to\infty}\left(\tfrac{3}{N}\sum_{k=1}^{N}1 + \tfrac{6}{N^2}\sum_{k=1}^{N}k + \tfrac{3}{N^3}\sum_{k=1}^{N}k^2\right)$$
    $$= \lim_{N\to\infty}\left(\frac{3}{N}(N) + \frac{6}{N^2}\frac{N(N+1)}{2} + \frac{3}{N^3}\frac{N(N+1)(2N+1)}{6}\right) = 3 + \tfrac{6}{2} + \frac{6}{6} = 7.$$

    Part (d): $\int_1^2 3x^2\,dx = \left[\ 3(\tfrac{1}{3})x^3\ \right]_1^2 = 2^3 - 1^3 = 7.$
14. $\int_2^5 \frac{1}{\sqrt{x^5}}\,dx = \int_2^5 x^{-\frac{5}{2}}\,dx = \left[\ \frac{1}{-3/2}x^{-\frac{3}{2}}\ \right]_2^5 = -\frac{2}{3}(5)^{-\frac{3}{2}} + \frac{2}{3}(2)^{-\frac{3}{2}}.$
15. $\left[\ -\frac{1}{2}e^{-x}\ \right]_0^1 == -\frac{1}{2}e^{-1} - \frac{1}{2}e^0 - \frac{1}{2e} - \frac{1}{2}.$
16. $\left[\ -\frac{1}{3}\ln|4-3x|\ \right]_1^3 = -\frac{1}{3}\ln|4-3(3)| - (-\frac{1}{3})\ln|4-3(1)| = -\frac{1}{3}\ln 5.$
17. $\left[\ \tan^{-1}x\ \right]_0^1 = \tan^{-1}1 - \tan^{-1}0 = \frac{\pi}{4}.$
22. $\left[\ \frac{x}{e^x}\ \right]_{-1}^1 = \frac{1}{e^1} - \frac{-1}{e^{-1}} = \frac{1}{e} + e.$
23. $\int_3^4 f'(x)\,dx = f(4) - f(3)$; from the grid we can approximate $\int_3^4 f'(x)\,dx \approx 8.5$, and we are given that $f(3) = 2$; therefore $8.5 \approx f(4) - 2$, so $f(4) \approx 10.5$.
26. $\int_{-2}^2 f'(x)\,dx \approx 9.5 - 7.5 = 2$ and $f(2) = 3$, so by FTC we have $2 \approx 3 - f(-2)$, thus $f(-2) \approx 1$.

Section 13.2 (continued)

27. Part (a): $\int_{-\pi}^{\pi} \cos x \, dx = \Big[\sin x \Big]_{-\pi}^{\pi} = \sin(\pi) - \sin(-\pi) = 0 - 0 = 0.$

Part (b): $-\int_{-\pi}^{-\frac{\pi}{2}} \cos x \, dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx - \int_{\frac{\pi}{2}}^{\pi} \cos x \, dx = 4.$

30. Part (a): $\int_1^4 \frac{3-2x}{x^2} \, dx = \int_1^4 (3x^{-2} - 2x^{-1}) \, dx = \Big[\frac{3}{-1}x^{-1} - 2\ln|x| \Big]_1^4 = \ldots = \frac{9}{4} - 2\ln 4.$

Part (b): (Note $\frac{3-2x}{x^2} = 0$ when $x = \frac{3}{2}$.)

$\int_1^{\frac{3}{2}} \frac{3-2x}{x^2} \, dx - \int_{\frac{3}{2}}^4 \frac{3-2x}{x^2} \, dx = \ldots = \frac{7}{4} - 4\ln(\frac{3}{2}) - 2\ln(4).$

32. (Note $x - 1 = x^2 - 2x - 1$ when $x = 0$ or $x = 3$.)

$\int_{-1}^{0} ((x^2 - 2x - 1) - (x - 1)) \, dx + \int_0^3 ((x - 1) - (x^2 - 2x - 1)) \, dx = \frac{19}{3}.$

38. $\frac{1}{2 - (-1)} \int_{-1}^{2} x^2 \sin(x^3 + 1) \, dx = \frac{1}{3} \Big[-\frac{1}{3}\cos(x^3 + 1) \Big]_{-1}^{2} = \ldots = -\frac{1}{9}\cos 9 + \frac{1}{9}.$

40. **Proof:** If $F(x) - G(x) = C$ then $F(x) = G(x) + C$, so:

$\Big[F(x) \Big]_a^b = \Big[G(x) + C \Big]_a^b = (G(b) + C) - (G(a) + C) = G(b) - G(a) = \Big[G(x) \Big]_a^b.$ ■

Section 13.3

3. TYPO: You can also assume that f is positive.
As x increases, the area $A(x)$ accumulated also increases, but at a rate that decreases (so the rate of change of the rate of change of $A(x)$ is negative, and thus $A(x)$ is concave down). More mathematically: $A''(x) = \frac{d}{dx}\left(\frac{d}{dx}\int_0^x f(t)\,dt\right) = \frac{d}{dx}(f(x)) = f'(x)$; therefore f decreasing $\Longrightarrow$ f' negative $\Longrightarrow$ A'' negative $\Longrightarrow$ A concave down.

6. Part (a): $A'(x)$ and $B'(x)$ are both equal to x^2, so $A(x)$ and $B(x)$ differ by a constant.
Part (b): $B(x) - A(x) = \int_3^x t^2\,dt - \int_0^x t^2\,dt = -\int_0^3 t^2\,dt$, which is a constant (an area).

8. $A(5) < A(0) < A(-1) < A(-2)$.

10. The graph of $A(x)$ should start at $(0,0)$, then increase on all of $[0,6]$. The graph should be concave up on $[0,3]$ and concave down on $[3,6]$ (so there is an IP at $x = 3$).

11. $A(x)$ is positive on approximately $[0,2]$ and negative on $[2,6]$. $A(x)$ is increasing on $[0,1]$ and $[5,6]$ and decreasing on $[1,5]$.

13. $f(x) + C$, $f(x)$.

14. $f(b) - f(a)$, 0.

15. $f(x) - f(a)$, $f(x)$.

16. $F(x) = A(g(x))$, where $A(x) = \int_3^x \sin t\,dt$ ("outside") and $g(x) = x^2$ ("inside").

19. f is continuous on $[1,5]$, so the MVT for integrals applies, and says that there is some $c \in (1,5)$ for which $\int_1^5 x(x-6)\,dx = c(c-6)(5-1)$. In other words, there is some $c \in (1,5)$ for which $f(c) = c(c-6)$ is equal to the average value $\frac{1}{5-1}\int_1^5 x(x-6)\,dx$ of the function f. You can compute this average value; it is equal to $-\frac{23}{3}$. Therefore there is some $c \in (1,5)$ such that $f(c) = -\frac{23}{3}$. We can solve the equation $f(c) = -\frac{23}{3}$ to find such a value of c; actually there are two such values (by the quadratic formula): $c \approx 1.8453$ and $x \approx 4.1547$.

23. True.

Section 13.3 (continued)

26. Part (a): $A(2)$ is the area under the graph of t^2+1 from $t=0$ to $t=2$.
Part (b): $A(2)=\int_0^2(t^2+1)\,dt=\left[\,\frac{1}{3}t^3+t\,\right]_0^2=\frac{8}{3}+2=\frac{14}{3}$, and $A(5)=\int_0^5(t^2+1)\,dt=\frac{40}{3}$.
Part (c): $A(x)=\int_0^x(t^2+1)\,dt=\left[\,\frac{1}{3}t^3+t\,\right]_0^x=\frac{1}{3}x^3+x$.

30. $\int_0^x \sin^2(3^t)\,dt$, $\int_2^x \sin^2(3^t)\,dt$, $\int_{-1}^x \sin^2(3^t)\,dt$.

33. $-e^{x^2+1}$.

34. $\cos(x^2)(2x)$.

37. $\int_x^{x+2}\sin(t^2)\,dt=\int_0^{x+2}\sin(t^2)\,dt-\int_0^x\sin(t^2)\,dt$, so the derivative is $\sin((x+2)^2)-\sin(x^2)$.

39. 0.

42. $\frac{d}{dx}(\frac{d}{dx}(-\int_1^{x^2}\ln|t|\,dt))=\frac{d}{dx}(-\ln|x^2|(2x))=-2\ln|x^2|-4$.

44. See the reading.

Section 14.1

4. Both integrals turn into $\int\frac{1}{u}\,du$ after a change of variables ($u=x^2+1$ in the first case, $u=\ln x$ in the second).

6. $\int 3x^2\sin(x^3)\,dx$, $\int\frac{\sin(\ln x)}{x}\,dx$, $\int\frac{\sin(\sqrt{x}+2)}{\sqrt{x}}\,dx$.

10. $du=(2x+1)\,dx$.

15. $\int_{-1}^5 u^2\,du=\frac{126}{3}=42$.
$\int_{x=-1}^{x=5}u^2\,du=\frac{1}{3}(5)^6-\frac{1}{3}(-1)^6=5208$.
$\int_{u(-1)}^{u(5)}u^2\,du=\int_1^{25}u^2\,du=\frac{1}{3}(25)^3-\frac{1}{3}(1)^3=5208$.

19. False.

21. True.

22. False.

26. Part (a): Choose $u=x^3+1\implies du=3x^2\,dx\implies\frac{1}{3}du=x^2\,dx$. Then:
$\int x^2(x^3+1)\,dx=\frac{1}{3}\int u\,du=\frac{1}{3}(\frac{1}{2}u^2)+C=\frac{1}{6}(x^3+1)^2+C$. (Don't forget you can check your answer by differentiating!)
Part (b): $\int x^2(x^3+1)\,dx=\int(x^5+x^2)\,dx=\frac{1}{6}x^6+\frac{1}{3}x^3+C$.
These differ by a constant: $\frac{1}{6}(x^3+1)^2=\frac{1}{6}(x^6+2x^3+1)=(\frac{1}{6}x^6=\frac{1}{3}x^3)+\frac{1}{6}$.

29. Part (a): Choose $u=\sqrt{x}+3$ to get $\int\frac{\sqrt{x}+3}{2\sqrt{x}}\,dx=\int u\,du=\frac{1}{2}(\sqrt{x}+3)^2+C$.
Part (b): Expand the integrand to get $\int(\frac{1}{2}+\frac{3}{2}x^{-\frac{1}{2}})\,dx=\frac{1}{2}x+3x^{\frac{1}{2}}+C$.
These are the same up to a constant: $\frac{1}{2}(\sqrt{x}+3)^2=(\frac{1}{2}x=3x^{\frac{1}{2}})+9$.

39. Choose $u=\cot x\implies du=\csc^2 x\,dx$ to get $\int u^5\,du=\frac{1}{6}u^6+C=\frac{1}{6}\cot^6 x+C$.

40. No substitution works; use algebra! (Multiply out the integrand.)

41. Choose $u=x^{\frac{5}{4}}\implies du=\frac{5}{4}x^{\frac{1}{4}}\,dx\implies\frac{4}{5}du=x^{\frac{1}{4}}\,dx$, to get $\frac{4}{5}\int\sin u\,du=\frac{4}{5}(-\cos u)+C=-\frac{4}{5}\cos(x^{\frac{5}{4}})+C$.

48. The substitution only becomes clear after some algebra; first rewrite $\ln\sqrt{x}=\ln(x^{\frac{1}{2}})=\frac{1}{2}\ln x$. Then choose $u=\ln x$ (so $du=\frac{1}{x}\,dx$) to get $\frac{1}{2}\int u\,du=\frac{1}{4}(\ln x)^2+C$.

49. Choose $u=2-e^x$ (so $-du=e^x\,dx$) to get $-\int\frac{1}{u}\,du=-\ln|2-e^x|+C$.

54. Choose $u=\frac{1}{x}\implies du=-x^{-2}\,dx\implies -du=\frac{1}{x^2}\,dx$ to get $\int sinu\,du=-\cos(\frac{1}{x})+C$.

57. Choose $u=x+1$, so $du=dx$ and $x=u-1$, to get $\int(u-1)\sqrt{u}\,du=\int(u^{\frac{3}{2}}-u^{\frac{1}{2}})\,du=\frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{2}{3}(x+1)^{\frac{3}{2}}+C$.

Section 14.1 (continued)

65. Choose $u = x - 1$ (so $du = dx$ and $x = u + 1$, thus $x^2 = (u+1)^2$) to get $\int (u+1)^2 u^{100}\, du = \int (u^{102} + 2u^{101} + u^{100})\, du = \frac{1}{103}(x-1)^{103} + \frac{2}{102}(x-1)^{102} + \frac{1}{101}(x-1)^{101} + C$.

66. Choose $u = x^2$ (that might take some trial and error to find!) so that $\frac{1}{2}du = x\, dx$ to get $\frac{1}{2}\int \frac{1}{1+u^2}\, du = \frac{1}{2}\tan^{-1} u + C = \frac{1}{2}\tan^{-1}(x^2) + C$.

69. In either case we will choose $u = x^2 + 1$ (so $\frac{1}{2}du = x\, dx$).
Part (a): $\int_0^3 x(x^2+1)^{\frac{1}{3}}\, dx = \frac{1}{2}\int_{x=0}^{x=3} u^{\frac{1}{3}}\, du = \frac{1}{2}\left[\ \frac{1}{4/3}u^{\frac{4}{3}}\ \right]_{x=0}^{x=3} = \frac{1}{2}\left[\ \frac{3}{4}(x^2+1)^{\frac{4}{3}}\ \right]_0^3 =$
$\frac{1}{2}(\frac{3}{4}(10)^{\frac{4}{3}} - \frac{3}{4}(1)^{\frac{4}{3}}) = \frac{3}{8}(10^{\frac{4}{3}} - 1)$.
Part (b): Now we change the limits of integration, using $u(0) = 0^2 + 1 = 1$ and $u(3) = 3^2 + 1 = 10$:
$\int_0^3 x(x^2+1)^{\frac{1}{3}}\, dx = \frac{1}{2}\int_1^{10} u^{\frac{1}{3}}\, du = \frac{1}{2}\left[\ \frac{4}{3}u^{\frac{4}{3}}\ \right]_1^{10} = \frac{1}{2}(\frac{3}{4}(10)^{\frac{4}{3}} - \frac{3}{4}(1)^{\frac{4}{3}}) = \frac{3}{8}(10^{\frac{4}{3}} - 1)$.

74. In either case choose $u = \ln x$ (so $du = \frac{1}{x}\, dx$).
Part (a): $\int_{x=1}^{x=2} u^{-\frac{1}{2}}\, du = \left[\ 2(\ln x)^{\frac{1}{2}}\ \right]_1^2 = 2(\ln 2)^{\frac{1}{2}}$.
Part (b): Now $u(1) = \ln 1 = 0$ and $u(2) = \ln 2$, so $\int_0^{\ln 2} u^{-\frac{1}{2}}\, du = \left[\ \frac{1}{1/2}u^{\frac{1}{2}}\ \right]_0^{\ln 2} = 2(\ln 2)^{\frac{1}{2}}$.

77. Part (a): $\int_{-1}^3 \frac{x}{x^2+1}\, dx = \frac{1}{2}(\ln 10 - \ln 2)$.
Part (b): (Note $f(x) = 0$ at $x = 0$.) $-\int_{-1}^0 \frac{x}{x^2+1}\, dx + \int_0^3 \frac{x}{x^2+1}\, dx = \frac{1}{2}(\ln 2 + \ln 10)$.
Part (c): (Note $f(x) = g(x)$ at $x = 0$, $x = \pm 1$.)
$\int_{-1}^0 \left(\frac{1}{2}x - \frac{x}{x^2+1}\right)\, dx + \int_0^1 \left(\frac{x}{x^2+1} - \frac{1}{2}x\right)\, dx + \int_1^3 \left(\frac{1}{2}x - \frac{x}{x^2+1}\right)\, dx = \frac{3}{2} + \frac{3}{2}\ln 2 - \frac{1}{2}\ln 10$.