Homework for Week 13 **Math 232 Spring 2002**

This homework will not be collected. It is your responsibility to do as many problems as necessary to understand the material (this includes doing extra problems if you need more practice). We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 14.2	1, 3, 9, 11, 12, 13, 18, 19, 20, 24, 25, 26, 27, 28, 31, 34, 35, 42, 45, 48, 49, 50, 54, 55, 59, 60, 65.
Section 14.3	4, 11, 14, 17, 19, 20, 21, 24, 27, 30, 31, 34, 35, 41, 48, 49, 50, 52, 54ab, 55ab.
Section 14.4	1, 2, 4, 15, 18, 24, 29, 31, 35, 36, 37, 38, 39, 43, 47, 48, 49, 55, 56, 69, 74.

Selected Hints and Answers

Caution: The hints and answers below are not necessarily full solutions. Many of them would not be considered complete on a quiz or test. **Answers are not provided for problems whose answers can be found in the reading or problems whose answers are easy to check using a calculator.**

Section 14.2

12. $du = 3\cos 3x\ dx$ and $dv = dx$. $\int u\ dv = \int \sin 3x\ dx$, while $\int v\ du = \int 3x\cos 3x\ dx$. The first integral is clearly easier, and the second integral can be rewritten in terms of the first using integration by parts.
13. Part (a): $\frac{d}{dx}(x\cos 2x) = \cos 2x - 2x\sin 2x$.
    Part (b): $\int(\cos 2x - 2x\sin 2x)\ dx = x\cos 2x + C$.
    Part (c): $\int(-2x\sin 2x)\ dx = x\cos 2x - \int \cos 2x\ dx$.
    (See how the integration by parts formula comes from the product rule for differentiation.)
18. $u = \ln x$, $v = -x^{-2}$.
20. False.
24. False.
25. **(a)** use parts with $u = \ln(x^3)$ and $dv = x^2\ dx$; **(b)** use substitution with $u = x^3$.
26. **(a)** use parts with $u = \ln x$ and $dv = \frac{1}{x}\ dx$; **(b)** use subsitution with $u = \ln x$.
27. **(a)** use parts with $u = x$ and $dv = (x+1)^{100}\ dx$; **(b)** use substitution with $u = x+1$ (with back-substitution).
34. Use parts with $u = x^2$ and $dv = e^{3x}\ dx$ to get $\frac{1}{3}x^2e^{3x} - \frac{2}{3}\int xe^{3x}\ dx$. Then use parts again to solve the new integral, with $u = x$ and $dv = e^{3x}\ dx$ to get $\frac{1}{3}x^2e^{3x} - \frac{2}{3}\left(\frac{1}{3}xe^{3x} - \frac{1}{3}\int e^{3x}\ dx\right) = \frac{1}{3}x^2e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C$.
35. $\frac{3}{2}e^{x^2} + C$.
42. Multiply out the integrand to get: $\int(x^2 - 2xe^x + e^{2x})\ dx = \int x^2\ dx - 2\int xe^x\ dx + \int e^{2x}\ dx$. The middle integral requires parts with $u = x$ and $dv = e^x\ dx$. After integrating we get: $\frac{1}{3}x^3 - 2(xe^x - e^x) + \frac{1}{2}e^{2x} + C$.
45. Use parts with $u = \sqrt{x}$ and $dv = \ln x\ dx$.
48. Use parts with $u = x$ and $dv = \csc^2 x\ dx$ to get $-x\cot x + \int \cot x\ dx$. Then use substitution with $u = \sin x$ (note that $\cot x = \frac{\cos x}{\sin x}$) to get $-x\cot x + \ln|\sin x| + C$.
49. Use parts three times, first with $u = x^3$ and $dv = \cos x\ dx$; then with $u = x^2$ and $dv = \sin x\ dx$; finally, with $u = x$ and $dv = \cos x\ dx$.

Section 14.2 (continued)

50. Use parts with $u = x^2$ and $dv = x\sin x^2$ (this is the largest piece of the integrand that you know how to integrate); you will have to use a substitution with $u = x^2$ to find v. After applying the integration by parts formula you should have $-\frac{1}{2}x^2\cos x^2 + \int x\cos x^2\ dx$. Now use substitution with $u = x^2$ to get $-\frac{1}{2}x^2\cos x^2 + \frac{1}{2}\sin x^2 + C$.

54. Use parts with $u = \tan^{-1} x$ and $dv = dx$ to get $x\tan^{-1} x - \int \frac{x}{1+x^2}\ dx$. Then use substitution with $u = 1 + x^2$ to get $x\tan^{-1} x - \frac{1}{2}\ln|1 + x^2| + C$.

55. Use parts twice, and solve an equation, to get $-\frac{1}{5}e^{2x}\cos x + \frac{2}{5}e^{2x}\sin x$ (see Example 6).

59. Use parts with $u = \ln x$ and $dv = dx$ to get $\left[\ x\ln x\ \right]_1^2 - \int_1^2\ dx = \left[\ x\ln x\ \right]_1^2 - \left[\ x\ \right]_1^2 = 2\ln 2 - 1$.

60. Use parts with $u = x$ and $dv = e^{-x}\ dx$ to get $\left[\ -xe^{-x}\ \right]_{-1}^1 + \int_{-1}^1 e^{-x}\ dx = \left[\ -xe^{-x}\ \right]_{-1}^1 + \left[\ -e^{-x}\ \right]_{-1}^1 = -\frac{2}{e}$.

65. Each of the problems below will involve using integration by parts with $u = x$ and $dv = \cos 2x\ dx$.
Part (a): $\int_0^{\frac{3\pi}{4}} x\cos 2x = -\frac{1}{4} - \frac{3\pi}{8} \approx -1.4281$.
Part (b): The solutions to $x\cos 2x = 0$ in the interval $[0, \frac{3\pi}{4}]$ are $x = 0$, $x = \frac{\pi}{4}$, and $x = \frac{3\pi}{4}$. Here we must compute $\int_0^{\frac{\pi}{4}} x\cos 2x\ dx - \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\cos 2x\ dx = (\frac{\pi}{8} - \frac{1}{4}) - (\frac{\pi}{2}) = \frac{5\pi}{8} - \frac{1}{4} \approx 1.7135$.
Part (c): The solutions to $x\cos 2x = -x$ on $[0, \frac{3\pi}{4}]$ are $x = 0$ and $x = \frac{\pi}{2}$. (However, the value of $f(x) = x\cos 2x$ is *always* greater than or equal to the value of $g(x) = -x$ on this interval, so we only need to calculate *one* definite integral.) The area between the two graphs is thus:
$\int_0^{\frac{3\pi}{4}} (x\cos 2x - (-x))\ dx = \int_0^{\frac{3\pi}{4}} x\cos 2x + \int_0^{\frac{3\pi}{4}} x\ dx = (-\frac{1}{4} - \frac{3\pi}{8}) + \frac{9\pi^2}{32} \approx 1.34773$.
Part (d): $\frac{1}{\frac{3\pi}{4} - 0}\int_0^{\frac{3\pi}{4}} x\cos 2x\ dx = \frac{4}{3\pi}(-\frac{1}{4} - \frac{3\pi}{8}) = -\frac{1}{3\pi} - \frac{1}{2} \approx -0.6061$.

Section 14.3

17. $\frac{1}{2}\int(1 - \cos 6x)\ dx = \frac{1}{2}x - \frac{1}{12}\sin(6x)$.

19. $\int(1-\sin^2 x)^2\cos x\ dx = \int\cos x\ dx - 2\int\sin^2 x\cos x\ dx + \int\sin^4 x\cos x\ dx = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C$.

20. $\int\frac{\sin 2x}{\cos 2x}\ dx = -\frac{1}{2}\ln|\cos 2x| + C$.

21. $\int\sec 4x\left(\frac{\sec 4x+\tan 4x}{\sec 4x+\tan 4x}\right)\ dx = \int\frac{\sec^2 4x+\sec 4x\tan 4x}{\sec 4x+\tan 4x}\ dx = \frac{1}{4}\ln|\sec 4x + \tan 4x| + C$.

24. $\int\cot^3 x(\csc^2 x - 1)\ dx = \int\cot^3 x\csc^2 x\ dx - \int\cot x(\csc^2 x - 1)\ dx = \int\cot^3 x\csc^2 x\ dx - \int\cot x\csc^2 x\ dx + \int\cot x\ dx = -\frac{1}{4}\cot^4 x + \frac{1}{2}\cot^2 x + \ln|\sin x| + C$ (use substitution with $u = \cot x$ for the first two integrals, and substitution with $u = \sin x$ for the last integral).

27. Use parts with $u = \csc^2 x$ and $dv = \csc^2 x\ dx$ to get $-\csc^2 x\cot x - 2\int\csc^2 x\cot^2 x\ dx$. Then use substitution with $u = \cot x$ to get $-\csc^2 x\cot x + \frac{2}{3}\cot^3 x + C$.

30. $\int\cos x(1 - \sin^2 x)\sin^4 x\ dx = \int(\sin^4 x - \sin^6 x)\cos x\ dx = \frac{1}{5}\sin^5 x - \frac{1}{7}\sin^7 x + C$.

31. $\int\left(\frac{1-\cos 6x}{2}\right)\left(\frac{1+\cos 6x}{2}\right)\ dx = \frac{1}{4}\int(1 - \cos^2 6x)\ dx = \frac{1}{4}\int(1 - \frac{1}{2}(1 + \cos(12x)))\ dx = \frac{1}{8}x - \frac{1}{96}\sin(12x) + C$.

34. $\int\sec^7 x(\sec x\tan x)\ dx = \frac{1}{8}\sec^8 x\ dx$.

35. $\int\sec^2 x(\sec^2 x - 1)^2(\sec x\tan x)\ dx = \int(\sec^6 x - 2\sec^4 x + \sec^2 x)(\sec x\tan x)\ dx = \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^2 x + C$.

41. $\int\left(\frac{\sin^2 x}{\cos^2 x}\right)\left(\frac{1}{\sin x}\right)\ dx = \int\frac{\sin x}{\cos^2 x}\ dx = -\int u^{-2}\ du = \frac{1}{\cos x} + C$.

48. $\int(\sec^2 x - 1)(\sec x\tan x)\ dx = \int\sec^2 x(\sec x\tan x)\ dx - \int\sec x\tan x\ dx = \frac{1}{3}\sec^3 x - \sec x + C$.
Check: $\frac{d}{dx}(\frac{1}{3}\sec^3 x - \sec x) = \frac{1}{3}(3)\sec^2 x(\sec x\tan x) - \sec x\tan x = (\sec^2 x - 1)(\sec x\tan x) = \tan^2 x(\sec x\tan x) = \sec x\tan^3 x$.

Section 14.3 (continued)

49. Part (a): Use $\int\left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)\,dx = \frac{1}{4}\int(1-\cos^2 2x)\,dx = \frac{1}{4}\int(1-\frac{1}{2}(1+\cos 4x))\,dx$.
Part (b): Use $\int(\sin x\cos x)^2\,dx = \int(\frac{1}{2}\sin 2x)^2\,dx = \frac{1}{4}\int\frac{1-\cos 4x}{2}\,dx$.

50. Part (a): $\int\tan^3 x\sec^2 x\,dx = \int u^3\,du$, with $u=\tan x$.
Part (b): $\int\sec x(\sec^2 x-1)(\sec x\tan x)dx = \int(\sec^3 x-\sec x)(\sec x\tan x)dx = \int(u^3-u)du$, with $u=\sec x$.

52. $\int_0^\pi \sin x(1-\cos^2 x)\cos^2 x\,dx = \int_0^\pi(\cos^2 x-\cos^4 x)\sin x\,dx = -\int_{x=0}^{x=\pi}(u^2-u^4)\,du$
$= -\left[\frac{1}{3}\cos^3 x-\frac{1}{5}\cos^5 x\right]_0^\pi = \frac{4}{15}$.

52. Part (b): $\int\frac{1}{2}(\sin(-x)+\sin(5x))\,dx = \frac{1}{2}\cos(-x)-\frac{1}{10}\cos(5x)+C$.

55. Part (a): $\int\sec^3 x\,dx = \frac{1}{2}\sec x\tan x+\frac{1}{2}\int\sec x\,dx = \frac{1}{2}\sec x\tan x+\ln|\sec x+\tan x|+C$.
$\int\sec^7 x\,dx = \frac{1}{6}\sec^5 x\tan x+\frac{5}{6}(\frac{1}{4}\sec^3 x\tan x+\frac{3}{4}(\frac{1}{2}\sec x\tan x+\frac{1}{2}\ln|\sec x+\tan x|))+C$.

Section 14.4

4. **(a)** $\int\frac{1}{\sqrt{x^2+1}}\,dx$; **(b)** $\int\frac{\sqrt{x^2-16}}{x}\,dx$; **(c)** $\int\sqrt{9-(x-2)^2}\,dx$; **(d)** $\int\frac{x}{x^2+1}\,dx$; **(e)** $\int\frac{1}{x^2-1}\,dx$.

29. $\sin u = \frac{x-5}{3}$, so make a triangle with angle u, opposite side $x-5$, and hypotenuse 3; then the remaining side is length $\sqrt{9-(x-5)^2}$, so $\tan^2 u = \frac{(x-5)^2}{9-(x-5)^2}$.

35. $4-6x-2x^2 = -2(x^2+3x-2) = -2((x^2+3x+9)-11) = -2((x+3)^2-11)$; for this type of quadratic you would use the trigonometric substitution $x+3=\sqrt{11}\sec u$.

36. Part (a): $u=\tan^{-1}x \Rightarrow du=\frac{1}{1+x^2}\,dx$ (and thus $x=\tan u$), which gives us the integral $\int\tan^2 u\,du = \int(\sec^2 u-1)\,du = \tan u-u+C = \tan(\tan^{-1}x)-\tan^{-1}x+C = x-\tan^{-1}x+C$.
Part (b): $x=\tan u \Rightarrow dx=\sec^2 u\,du$ (and thus $u=\tan^{-1}x$) gives us the integral $\int\frac{\tan^2 u}{1+\tan^2 u}\sec^2 u\,du = \int\tan^2 u\,du = \ldots$ same as above $\ldots = x-\tan^{-1}x+C$.

38. Part (a): $x=3\tan u \Rightarrow dx=3\sec^2 u\,du$ (and $u=\tan^{-1}\frac{x}{3}$) gives us the integral:
$\int\frac{1}{9\tan^2 u+9}(3\sec^2 u)\,du = \frac{1}{3}\int du = \frac{1}{3}u+C = \frac{1}{3}\tan^{-1}\frac{x}{3}+C$.
Part (b): $\int\frac{1}{x^2+9}\,dx = \frac{1}{9}\int\frac{1}{(\frac{x}{3})^2+1}\,dx = \frac{1}{9}(3\tan^{-1}\frac{x}{3})+C = \frac{1}{3}\tan^{-1}\frac{x}{3}+C$.

39. Part (a): $u=4+x^2 \Rightarrow du=2x\,dx \Rightarrow \frac{1}{2}\,du=x\,dx$. This gives the integral:
$\frac{1}{2}\int\frac{1}{u}\,du = \frac{1}{2}\ln|4+x^2|+C$.
Part (b): $x=2\tan u \Rightarrow dx=2\sec^2 u\,du$ (and $u=\tan^{-1}\frac{x}{2}$) gives us the integral:
$\int\frac{2\tan u}{4+4\tan^2 u}(2\sec^2 u)\,du = \int\tan u\,du = \int\frac{\sin u}{\cos u}\,du = -\ln|\cos u|+C =$
$-\ln|\cos(\tan^{-1}\frac{x}{2})|+C = -\ln|\frac{2}{\sqrt{x^2+4}}|+C$. (The last step follows from making a triangle.)
(Note that these answers do differ by a constant, since $-\ln|\frac{2}{\sqrt{x^2+4}}| = -\ln 2+\frac{1}{2}\ln(x^2+4)$.

43. $x=\sqrt{3}\sin u \Rightarrow dx=\sqrt{3}\cos u\,du$ (and $u=\sin^{-1}\frac{x}{\sqrt{3}}$) gives the integral:
$\int\frac{1}{\sqrt{3-3\sin^2 u}}(\sqrt{3}\cos u)\,du = \int du = u+C = \sin^{-1}\frac{x}{\sqrt{3}}+C$.
Check: $\frac{d}{dx}(\sin^{-1}\frac{x}{\sqrt{3}}) = \frac{1}{\sqrt{1-(\frac{x}{\sqrt{3}})^2}}(\frac{1}{\sqrt{3}}) = \frac{1}{\sqrt{3}\sqrt{1-\frac{x^2}{3}}} = \frac{1}{\sqrt{3-x^2}}$.

47. $x=2\sin u \Rightarrow dx=2\cos u\,du$ (and $u=\sin^{-1}\frac{x}{2}$) gives the integral
$\int\frac{\sqrt{4-4\sin^2 u}}{4\sin^2 u}(2\cos u)\,du = \int\frac{\cos^2 u}{\sin^2 u}\,du = \int\cot^2 u\,du = \int(\csc^2 u-1)\,du$
$= -\cot(\sin^{-1}\frac{x}{2})-\sin^{-1}\frac{x}{2}+C = -\frac{\sqrt{4-x^2}}{x}-\sin^{-1}\frac{x}{2}+C$.

48. This one is tricky because it involves a trignonometric with secant. Choose $x=\sec u \Rightarrow dx=\sec u\tan u\,du$ (and $u=\sec^{-1}x$). Then we have
$\int\frac{\sqrt{\sec^2 u-1}}{\sec u}(\sec u\tan u)\,du = \int\sqrt{\tan^2 u}\tan u\,du$. At this point we must split into cases...
If $x>1$ (and thus $0\le u<\frac{\pi}{2}$), we have the integral $\int\tan^2 u\,du = \int(\sec^2 u-1)\,du =$ $\tan u-u+C = \tan(\sec^{-1}x)-\sec^{-1}x+C = \sqrt{x^2-1}-\sec^{-1}x+C$ (the last equality used a triangle in the first quadrant of the unit circle).
If $x<-1$ (and thus $\frac{\pi}{2}<u\le\pi$), we have $-\int\tan^2 u\,du = -\tan(\sec^{-1}x)+\sec^{-1}x+C$
$=\sqrt{x^2-1}+\sec^{-1}x+C$ (using a triangle in the fourth quadrant).

Section 14.4 (continued)

49. $u = x^2+1 \Rightarrow du = 2x\ dx \Rightarrow \frac{1}{2}\ du = x\ dx$ gives us $\int \frac{x}{\sqrt{x^2-1}}\ dx = \frac{1}{2}\int u^{-\frac{1}{2}}\ du = (x^2-1)^{\frac{1}{2}} + C$.

55. $x = \sin u \Rightarrow dx = \cos u\ du$ gives $\int (1-\sin^2 u)^{-\frac{3}{2}} \cos u\ du = \int \frac{1}{\cos^2 u}\ du = \int \sec^2 u\ du = \tan u + C = \tan(\sin^{-1} x) + C = \frac{x}{\sqrt{1-x^2}} + C$.

56. Complete the square first: $\int \frac{1}{x^2-4x+13}\ dx = \frac{1}{(x-2)^2+9}\ dx$. Now $x-2 = 3\tan u \Rightarrow dx = 3\sec^2 u\ du$ (and $u = \tan^{-1}(\frac{x-2}{3})$) gives us the integral: $\int \frac{1}{(3\tan u)^2+9}(3\sec^2 u)\ du = \frac{1}{3}\int du = \frac{1}{3}\tan^{-1}(\frac{x-2}{3}) + C$.

69. $x = 2\tan u \Rightarrow dx = 2\sec^2 u\ du$ (and $u = \tan^{-1}\frac{x}{2}$) gives $8\int_{x=0}^{x=4} \tan u \sec^3 u\ du = 8\int_{x=0}^{x=4} \sec^2 u(\sec u \tan u)\ du$. Now use substitution with $w = \sec u$ (so $dw = \sec u \tan u\ du$) to get $8\int_{x=0}^{x=4} w^2\ dw = 8\left[\ \frac{1}{3}w^3\ \right]_{x=0}^{x=4} = 8\left[\ \frac{1}{3}\sec^3 u\ \right]_{x=0}^{x=4} = 8\left[\ \frac{1}{3}\sec^3(\tan^{-1}\frac{x}{2})\ \right]_0^4 = 8\left[\ \frac{1}{3}(\frac{\sqrt{x^2+4}}{2})^3\ \right]_0^4 = \left[\ \frac{1}{3}(x^2+4)^{\frac{3}{2}}\ \right]_0^4 = \frac{1}{3}(20)^{\frac{3}{2}} - \frac{1}{3}(4)^{\frac{3}{2}}$.

74. Part (a): $2\int_{-r}^{r} \sqrt{r^2-x^2}\ dx$.

Part (b): $x = r\sin u \Rightarrow dx = r\cos u\ du$ (and $u = \sin^{-1}(\frac{x}{r})$). When $x = r$ we have $u = \sin^{-1}\frac{r}{r} = \sin^{-1} 1 = \frac{\pi}{2}$, and when $x = -r$ we have $u = \sin^{-1}\frac{-r}{r} = \sin^{-1}(-1) = -\frac{\pi}{2}$.
This gives us the new integral:

$$2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{r^2 - r^2\sin^2 u}\ (r\cos u)\ du = 2r^2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^2 u\ du = 2r^2(\tfrac{1}{2})\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2u)\ du$$

$$= r^2\left[\ u + \tfrac{1}{2}\sin 2u\ \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = r^2((\tfrac{\pi}{2} + \tfrac{1}{2}\sin\pi) - (-\tfrac{\pi}{2} + \tfrac{1}{2}\sin(-\pi))) = \pi r^2.$$