Homework for Week 14 **Math 232 Spring 2002**

This homework will not be collected. It is your responsibility to do as many problems as necessary to understand the material (this includes doing extra problems if you need more practice). We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 15.1 1, 2, 3, 7, 8, 10, 16, 19, 21, 24, 25, 31*, 37, 40, 44, 45, 47, 51, 53, 55.

Section 15.2 5, 6, 8, 10, 13, 14, 18, 25, 27, 28, 30, 31, 33, 35, 37, 39, 41, 44, 48, 50, 56.

Selected Hints and Answers

Caution: The hints and answers below are not necessarily full solutions. Many of them would not be considered complete on a quiz or test. **Answers are not provided for problems whose answers can be found in the reading or problems whose answers are easy to check using a calculator.**

Section 15.1

7. Part (d): Hint: What is the shortest distance between two points?

10. Both arc lengths are given by the definite integral $\int_0^2 \sqrt{1+4x^2}\,dx$, so they are equal. This makes sense because $f(x) = x^2 - 3$ and $g(x) = 5 - x^2$ differ by a vertical flip and shift only; these transformations should not affect the arc length.

16. The top part of the circle is given by the equation $y = \sqrt{25 - x^2}$. Since $y' = \frac{x}{\sqrt{25-x^2}}$, the circumference of the circle (twice the arc length of the top of the circle) is given by the definite integral $2\int \sqrt{1 + (\frac{x}{\sqrt{25-x^2}})^2}\,dx$.

19. If $\sqrt{1+9e^{6x}} = \sqrt{1+(f'(x))^2}$ then $(f'(x))^2 = 9e^{6x}$. Therefore $f'(x) = 3e^{3x}$, so $f(x) = e^{3x}$ (actually any function $e^{3x} + C$ will work). Therefore the definite integral given represents the arc length of $f(x) = e^{3x}$ on the interval $[-2, 5]$.

25. $\sqrt{(-1-(-2))^2 + (-1-(-8))^2} + \sqrt{(0-(-1))^2 + (0-(-1))^2} + \sqrt{(1-0)^2 + (1-0)^2} + \sqrt{(2-1)^2 + (8-1)^2} \approx 16.97$.

31. TYPO: The "minus" should be a "plus" in the given sigma notation formula.
$\sqrt{1+(\frac{7}{1})^2}\,(1) + \sqrt{1+(\frac{1}{1})^2}\,(1) + \sqrt{1+(\frac{1}{1})^2}\,(1) + \sqrt{1+(\frac{7}{1})^2}\,(1) \approx 16.97$.
Here $\Delta x = 1$ and $x_k = -2 + k$, so $\Delta y_k = f(x_k) - f(x_{k-1}) = (-2+k)^3 + (-2+(k-1))^3$.
(This gives us $\Delta y_1 = 7$, $\Delta y_2 = 1$, $\Delta y_3 = 1$, and $\Delta y_4 = 7$.) Therefore in sigma notation, the sum is: $\displaystyle\sum_{k=1}^{4} \sqrt{1 + \left(\frac{(-2+k)^3 - (-2+(k-1))^3}{1}\right)^2}\,(1)$.

37. $\displaystyle\int_{-1}^{4} \sqrt{10}\,dx = [\,\sqrt{10}x\,]_{-1}^{4} = 5\sqrt{10}$.

40. $\displaystyle\int_{1}^{3} \sqrt{1+9x}\,dx = [\,\frac{2}{27}(1+9x)^{\frac{3}{2}}\,]_1^3 = \frac{2}{27}(28)^{\frac{3}{2}} - \frac{2}{27}(10)^{\frac{3}{2}}$.

44. $\displaystyle\int_{-3}^{3} \frac{3}{\sqrt{9-x^2}}\,dx = [\,3\sin^{-1}(\frac{x}{3})\,]_{-3}^{3} = 3\pi$.

45. $\displaystyle\frac{1}{2}\int_0^1 \frac{1+x}{\sqrt{x}}\,dx = \frac{1}{2}\,[\,2x^{\frac{1}{2}} + \frac{2}{3}x^{\frac{3}{2}}\,]_0^1 = \frac{4}{3}$.

47. $\displaystyle\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \csc x\,dx = [\,-\ln|\csc x + \cot x|\,]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} = \ln|-1+\sqrt{2}| - \ln|1+\sqrt{2}|$.

Section 15.1 (continued)

53. Part (a): The distance from $(a, f(a)) = (a, ma+c)$ to $(b, f(b)) = (b, mb+c)$ is given by $\sqrt{(b-a)^2 + ((mb+c)-(ma+c))^2} = \sqrt{(b-a)^2 + m^2(b-a)^2} = (b-a)\sqrt{1+m^2}$. (Since the graph of $f(x) = mx + c$ is a line, this is the exact arc length.)
Part (b): If $f(x) = mx + c$ then $f'(x) = m$ (a constant), so the arc length is given by $\int_a^b \sqrt{1+m^2}\, dx = \left[\ \sqrt{1+m^2}\ \right]_a^b = \sqrt{1+m^2}(b-a)$.

55. If $f(x) = \sqrt{r^2 - x^2}$ (the top half of the circle), then $f'(x) = -\frac{x}{\sqrt{r^2-x^2}}$. Thus the circumference of the circle is (twice the arc length of the top half): $2\int_{-r}^{r} \sqrt{1 + (-\frac{x}{\sqrt{r^2-x^2}})^2}\, dx = 2\int_{-r}^{r} \frac{r}{\sqrt{r^2-x^2}}\, dx$. Using trigonometric substitution with $x = r\sin u$ (and changing the limits of integration accordingly), this is equal to: $2r\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 1\, du = 2r\ \left[\ u\ \right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2r(\frac{\pi}{2} - (-\frac{\pi}{2})) = 2\pi r$.

Section 15.2

5. Part (a): $\Delta y = 1$ (width of each vertical subinterval), and $y_0 = 0$, $y_1 = 1$, $y_2 = 2$, $y_3 = 3$, $y_4 = 4$ (heights on the y-axis).
Part (b): One choice is $y_1^* = 0.5$, $y_2^* = 1.5$, $y_3^* = 2.5$, and $y_4^* = 3.5$ (the midpoints of each vertical subinterval).
Part (c): $f(x) = x^2$ so $f^{-1}(y) = \sqrt{y}$; thus $f^{-1}(y_1^*) = \sqrt{0.5} \approx 0.707$, $f^{-1}(y_2^*) = \sqrt{1.5} \approx 1.225$, $f^{-1}(y_3^*) = \sqrt{2.5} \approx 1.581$, and $f^{-1}(y_4^*) = \sqrt{3.5} \approx 1.87$. (These values are locations on the x-axis.)
Part (d): The disc at y_2^* has volume $\pi(f^{-1}(y_2^*))^2\ \Delta y = \pi(\sqrt{1.5})^2(1) = 1.5\pi$.
Part (e): The disc at y_k^* has volume $\pi(f^{-1}(y_k^*))^2\ \Delta y = \pi(\sqrt{y_k^*})^2(1) = y_k^*\,\pi$.

6. Part (a): $\Delta x = \frac{1}{2}$ (the width of each subinterval), and $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = 1.5$, $x_4 = 2$ (locations on the x-axis).
Part (b): One choice is $x_1^* = 0.25$, $x_2^* = 0.75$, $x_3^* = 1.25$, and $x_4^* = 1.75$ (the midpoints of each subinterval).
Part (c): $f(x_1^*) = (0.25)^2 = 0.0625$, $f(x_2^*) = (0.75)^2 = 0.5625$, $f(x_3^*) = (1.25)^2 = 1.5625$, and $f(x_4^*) = 3.0625$. (These values are heights, *i.e.* locations on the y-axis.)
Part (d): The washer at x_2^* has volume
$\pi(4)^2 - \pi(f(x_2^*))^2 = 16\pi - \pi((0.75)^2)^2 \approx \pi(16 - 0.316) \approx 49.27$.
Part (e): The washer at x_k^* has volume
$\pi(4)^2 - \pi(f(x_k^*))^2 = 16\pi - \pi((x_k^*)^2)^2 = \pi(16 - (x_k^*)^4)$.

8. $\pi\int_2^3 (1+x)^2\, dx$; this is the volume of the solid of revolution obtained by rotating the region between the graph of $f(x) = 1 + x$ and the x-axis on $[2, 3]$ around the x-axis.

10. $\pi\int_0^2 (x^2 - 4)\, dx$; this is the volume of the solid of revolution obtained by rotating the region between the graph of $y = x^2$ and $y = 2$ on $[0, 2]$ around the x-axis.

13. The region between $f(x) = x - 1$ and the x-axis on $[1, 3]$, rotated around the x-axis.

14. The region between the graph of $f(x) = 2x + 1$ and the line $y = 5$ on the x-interval $[0, 2]$, rotated about the y-axis.

18. $\pi\int_0^{\sqrt{3}} ((3 + 4x - x^2)^2 - (4x)^2)\, dx + \pi\int_{\sqrt{3}}^4 ((4x)^2 - (3 + 4x - x^2)^2)\, dx$.

25. The first integral represents the volume of the solid of revolution obtained by rotating the region between $y = x^2$ and the x-axis on $[0, 2]$ around the x-axis. The second integral represents the volume of the solid of revolution obtained by rotating the region between $y = x^2$ and the line $y = 4$ on the x-interval $[0, 2]$ around the y-axis. The second solid of revolution is larger (the region that is rotated is larger, because the graph of $y = x^2$ is concave up; look at the graph).

Section 15.2 (continued)

27. Part (a): The volume in question can be obtained by subtracting 8π from the volume of a cylinder of radius $\sqrt{b}$ and "height" b (draw a picture); thus the volume is $\pi(\sqrt{b})^2(b) - 8\pi = \pi(b^2 - 8)$.
Part (b): $\pi\int_0^b (\sqrt{x})^2\, dx = 8\pi \Rightarrow \pi\left[\, \frac{1}{2}x^2 \,\right]_0^b = 8\pi \Rightarrow \pi(\frac{1}{2}b^2 - 0) = 8\pi \Rightarrow b = 4$.

28. One example is the region bounded by the graph of $y = \sqrt{x}$, the graph of $y = x - 2$, and the x-axis, rotated around the x-axis. (Discs are needed on $[0, 2]$, but washers are needed on $[2, 4]$.)

30. One example is the region bounded by the graph of $y = x^3 + x + 1$ and the x-axis on $[0, 4]$, rotated around the y-axis. To use discs to find the volume of this region, you would have to find the inverse of $y = x^3 + x + 1$ (why?), which we don't know how to do.

31. Part (a): $\Delta x = 1$; using right hand side of each subinterval we get:
$\pi(\sqrt{1})^2(1) + \pi(\sqrt{2})^2(1) + \pi(\sqrt{3})^2(1) + \pi(\sqrt{4})^2(1) = 10\pi \approx 31.4159$.
Part (b): Since $y = \sqrt{x}$ we have $x = y^2$. Here $\Delta y = 0.5$; using the "top" of each vertical subinterval we get:
$\pi(16 - ((0.5)^2)^2)(0.5) + \pi(16 - ((1)^2)^2)(0.5) + \pi(16 - ((1.5)^2)^2)(0.5) + \pi(16 - ((2)^2)^2)(0.5) \approx 65.7771$.
Part (c): With $\Delta x = 1$ and the right hand side of each subinterval we have:
$\pi(25 - (5 - \sqrt{1})^2)(1) + \pi(25 - (5 - \sqrt{2})^2)(1) + \pi(25 - (5 - \sqrt{3})^2)(1) + \pi(25 - (5 - \sqrt{4})^2)(1) \approx 161.675$.
Part (d): With $\Delta y = 0.5$ and the "top" of each vertical subinterval, we have:
$\pi(49 - (3 + (0.5)^2)^2)(0.5) + \pi(49 - (3 + (1)^2)^2)(0.5) + \pi(49 - (3 + (1.5)^2)^2)(0.5) + \pi(49 - (3 + (2)^2)^2)(0.5) \approx 145.888$.

35. Part (a): Volume of disc at x_k^* is $\pi(\sqrt{x_k^*})^2\, \Delta x = \pi\, x_k^*\, \Delta x$.
Part (b): Volume of washer at y_k^* is $\pi(4^2 - ((y_k^*)^2)^2)\, \Delta y = \pi(16 - (y_k^*)^4)\, \Delta y$.
Part (c): Volume of washer at x_k^* is $\pi(25 - (5 - \sqrt{x_k^*})^2)\, \Delta x$.
Part (d): Volume of washer at y_k^* is $\pi(49 - (3 + (y_k^*)^2)^2)\, \Delta y$.

39. Part (a): $\pi\int_0^4 x\, dx = 8\pi \approx 25.1327$.
Part (b): $\pi\int_0^2 (16 - y^4)\, dy = \frac{128\pi}{5} \approx 80.4248$.
Part (c): $\pi\int_0^4 (25 - (5 - \sqrt{x})^2)\, dx = \frac{136\pi}{3} \approx 142.419$ (multiply out integrand).
Part (d): $\pi\int_0^2 (49 - (3 + y^2)^2)\, dy = \frac{288\pi}{5} \approx 180.956$ (multiply out integrand).

48. Part (a): $\pi\int_2^5 (x - 2)^2\, dx = \pi\int_2^5 (x^2 - 4x + 4)\, dx = 9\pi$.
Part (b): $\pi\int_0^3 (25 - (y + 2)^2)\, dy = \pi\int_0^3 (-y^2 - 4y + 21)\, dy = 36\pi$.
Part (c): $\pi\int_2^5 (9 - (3 - (x - 2))^2)\, dx = \pi\int_2^5 (-x^2 + 10x - 16)\, dx = 18\pi$.
Part (d): $\pi\int_0^3 (9 - ((y + 2) - 2)^2)\, dy = \pi\int_0^3 (9 - y^2)\, dy = 18\pi$.
Part (e): $\pi\int_2^5 ((2 + (x - 2))^2 - 4)\, dx = \pi\int_2^5 (x^2 - 4)\, dx = 27\pi$.
Part (f): $\pi\int_0^3 ((6 - (y + 2))^2 - 1)\, dy = \pi\int_0^3 (y^2 - 8y + 15)\, dy = 18\pi$.

50. See Example 5; in this case the cone is formed by rotating the region between the graph of $f(x) = \frac{5}{3}x$ and the line $y = 5$ on the x-interval $[0, 3]$ around the y-axis.

56. A sphere of radius r can be obtained by rotating the region under the graph of $y = \sqrt{r^2 - x^2}$ on $[-r, r]$ around the x-axis. The volume of this solid is given by the definite integral $\pi\int_{-r}^{r} (\sqrt{r^2 - x^2})^2\, dx = \pi\int_{-r}^{r} (r^2 - x^2)\, dx = \pi\left[\, r^2x - \frac{1}{3}x^3 \,\right]_{-r}^{r} = \pi((r^2(r) - \frac{1}{3}r^3) - (r^2(-r) - \frac{1}{3}(-r)^3)) = \pi(r^3 - \frac{1}{3}r^3 + r^3 - \frac{1}{3}r^3) = \pi r^3(2 - \frac{2}{3}) = \frac{4}{3}\pi r^3$.