Homework for Week 5 **Math 232 Spring 2001**

This homework will not be collected. It is your responsibility to do as many problems as necessary to understand the material (this includes doing extra problems if you need more practice). We recommend that you read each section before attempting any exercises. Next week's quiz will be a subset of the problems below.

Section 9.1 1, 3, 4, 13, 15, 19, 24, 28, 32, 34, 40, 42, 46, 49, 50, 51, 55, 59, 64, 66.

Section 9.2 3, 4, 5, 8, 11, 12, 14, 15, 20, 26, 30, 32, 33, 34, 38, 40, 43, 48, 49, 51, 56, 58, 60.

Section 9.3 1, 2, 3, 4, 5, 6, 8, 11, 13, 16, 19, 23, 24, 25, 27, 28, 35, 37, 40, 43, 47, 51.

Selected Hints and Answers

Caution: The hints and answers below are not necessarily full solutions. Many of them would not be considered complete on a quiz or test. **Answers are not provided for problems whose answers can be found in the reading or problems whose answers are easy to check using a calculator.**

Section 9.1

3. (a) Domain $(0,\infty)$, range $(-\infty,\infty)$; (b) domain $(3,\infty)$, range $(-\infty,\infty)$.

15. Hint: $f(x) = 2\ln x$, and $g(x) = -\ln x$.

19. False.

24. $\log_2 = \log_3 \implies \frac{\ln x}{\ln 2} = \frac{\ln x}{\ln 3} \implies (\ln 3)(\ln x) = (\ln 2)(\ln x)$
$\implies (\ln x)(\ln 3 - \ln 2) = 0 \implies \ln x = 0 \implies x = 1.$

32. $\ln(\frac{1}{e^2}) = \ln(e^{-2}) = -2\ln e = -2(1) = -2.$

40. $4\log_2 6 - 2\log_2 9 = \log_2(\frac{6^4}{9^2}) = \log_2 16 = 4.$

49. $(-\infty,-1)\cup(2,\infty)$.

50. We have to have $\ln(x-2) \neq 0$ (so $x-2 \neq 1$), $x-2 > 0$, and $x+1 > 0$; put this together to get $(2,3)\cup(3,\infty)$.

51. Stretch $y = \log_4 x$ vertically by a factor of three, then shift down by 8 units.

55. $\log_x 8 = 3 \implies x^3 = 8 \implies x = 8^{\frac{1}{3}}$.

59. $\frac{4^{x-1}}{2^x} = 6 \implies \frac{1}{4}(2^x) = 6 \implies 2^x = 24 \implies x = \log_2(24)$ is the exact answer; the approximate answer is $x = \log_2(24) = \frac{\ln 2}{\ln 24} \approx 0.2181$.

66. **Proof:** $\log_b(\frac{x}{y}) = \log_b(x\cdot y^{-1}) = \log_b x + \log_b(y^{-1})$ (by part (b) of Theorem 3)
$= \log_b x - \log_b y.$ ■ (by part (a) of Theorem 3)

Section 9.2

3. Part (a): $\lim_{x\to\infty}\frac{\ln(x^5)}{\ln(x^4)} \overset{\text{L'H}}{=} \lim_{x\to\infty}\frac{(\frac{5x^4}{x^5})}{(\frac{4x^3}{x^4})} = \lim_{x\to\infty}\frac{5}{4} = \frac{5}{4}.$

Part (b): $\lim_{x\to\infty}\frac{\ln(x^5)}{\ln(x^4)} = \lim_{x\to\infty}\frac{5\ln x}{4\ln x} = \lim_{x\to\infty}\frac{5}{4} = \frac{5}{4}.$

4. Part (a):
$$\frac{d}{dx}\left(\frac{\ln(x^5)}{\ln(x^4)}\right) = \frac{\frac{5x^4}{x^5}\ln(x^4) - \ln(x^5)\frac{4x^3}{x^4}}{(\ln(x^4))^2} = \frac{5\ln(x^4) - 4\ln(x^5)}{x(\ln(x^4))^2} = \frac{20\ln x - 20\ln x}{x(\ln(x^4))^2} = 0.$$

Part (b): $\frac{d}{dx}\left(\frac{\ln(x^5)}{\ln(x^4)}\right) = \frac{d}{dx}\left(\frac{5\ln x}{4\ln x}\right) = \frac{d}{dx}\left(\frac{5}{4}\right) = 0.$

Section 9.2 (continued)

12. 0.

14. By educated guessing, $f(x) = \frac{3}{2}\ln(2x+1) + C$; since $f(2) = 6$ we have $C = 6 - \frac{3}{2}\ln 5$.

15. By educated guessing, $f(x) = \ln(x^2+1) + C$; since $f(0) = -2$ we have $C = -2$.

20. $\displaystyle\lim_{x\to 0^+} \frac{x}{\log_2 x} \to \frac{0}{-\infty} \to 0.$

26. $\displaystyle\lim_{x\to 2^+} \frac{\ln(x-2)}{\ln(x^2-4)} \overset{\text{L'H}}{=} \lim_{x\to 2^+} \frac{(\frac{1}{x-2})}{(\frac{2x}{x^2-4})} = \lim_{x\to 2^+} \frac{x^2-4}{2x(x-2)} = \lim_{x\to 2^+} \frac{(x-2)(x+2)}{2x(x-2)} = \lim_{x\to 2^+} \frac{4}{4} = 1.$

30. $\displaystyle\lim_{x\to\infty} \ln\left(\frac{e^x}{1+e^x}\right) = \ln 1 = 0.$

32. $\frac{d}{dx}(\ln(\sqrt{x})) = \frac{d}{dx}(\frac{1}{2}\ln x) = \frac{1}{2x}$, so $x = 0$ is the only CP of $f(x)$.

33. $\frac{d}{dx}(\sqrt{\ln x}) = \frac{1}{2}(\ln x)^{-\frac{1}{2}}(\frac{1}{x}) = \frac{1}{2x\sqrt{\ln x}}$, so $x = 0$ and $x = 1$ are the CPs of $f(x)$.

34. $\frac{d}{dx}(\log_2(3^x)) = \frac{1}{(\ln 2)3^x}(\ln 3)3^x = \frac{(\ln 3)3^x}{(\ln 2)3^x}$, so $f(x)$ has no CPs.

38. After a bunch of algebra, $f'(x) = \frac{(x-1)(x+3)}{x(x^2+x+1)}$, so $x = 1$, $x = -3$, and $x = 0$ are the CPs.

40. Hint: Use this algebra first: $x^2\log_2(x2^x) = x^2(\log_2 x + \log_2(2^x)) = x^2(\log_2 x + x)$. Now use the product rule.

43. $\frac{d}{dx}(x^2\ln(\ln x)) = 2x\ln(\ln x) + x^2\frac{1}{\ln x}\cdot\frac{1}{x}$.

48. $\displaystyle\lim_{x\to\infty} \frac{x+2}{\log_3(1000x^{50})} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{1}{\frac{1}{(\ln 3)1000x^{50}}(1000(50)x^{49})} = \lim_{x\to\infty} \frac{1000x^{50}(\ln 3)}{50x^{49}} = \infty;$
thus g dominates f.

49. $\displaystyle\lim_{x\to\infty} \frac{\ln(x^{100}+x^{50}+200)}{\ln x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{\left(\frac{100x^{99}+50x^{49}}{x^{100}+x^{50}+200}\right)}{(\frac{1}{x})} = \lim_{x\to\infty} \frac{100x^{100}+50x^{50}}{x^{100}+x^{50}+200} = 100;$
since the limit is not ∞ or 0, neither function domainates.

51. $\displaystyle\lim_{x\to\infty} \frac{\log_{30} x}{\log_2 x} \overset{\text{L'H}}{=} \lim_{x\to\infty} \frac{(\frac{1}{(\ln 30)x})}{(\frac{1}{(\ln 2)x})} = \lim_{x\to\infty} \frac{\ln 2}{\ln 30} = \frac{\ln 2}{\ln 30};$
since the limit is not ∞ or 0, neither function domainates.

56. • First notice that $f(x)$ can be written $f(x) = -\ln(x^2-1)$.
• The domain of $f(x)$ is where $x^2 - 1 > 0$, *i.e.* $(-\infty,-1)\cup(1,\infty)$.
• $f(x) = 0$ if $x^2 - 1 = 1$, *i.e.* if $x = \pm\sqrt{2}$. The number line for $f(x)$ is marked at $-\sqrt{2}, -1, 1, \sqrt{2}$ with the subintervals marked as $-$, $+$, DNE (from $x = -1$ to $x = 1$), $+$, $-$.
• $f'(x) = \frac{-2x}{x^2-1}$, which is zero at $x = 0$ (which is not in the domain of f), and DNE at $x = \pm 1$. $f'(x)$ is positive on $(-\infty,-1)$ and negative on $(1,\infty)$.
• $f''(x) = \frac{2(x^2+2)}{(x^2-1)^2}$ is never zero, DNE at ± 1. $f''(x)$ is positive on $(-\infty,-1)$ and on $(1,\infty)$.
• $\displaystyle\lim_{x\to -1^-}(-\ln(x^2-1)) \to -(-\infty) \to \infty$, and $\displaystyle\lim_{x\to 1^+}(-\ln(x^2-1)) \to -(-\infty) \to \infty$.
• $\displaystyle\lim_{x\to -\infty}(-\ln(x^2-1)) \to -\infty$, and $\displaystyle\lim_{x\to\infty}(-\ln(x^2-1)) \to \infty$.
• In summary, the graph of $f(x)$ is concave up and increasing on $(-\infty,-1)$, concave up and decreasing on $(1,\infty)$, has roots at $x = \pm\sqrt{2}$, has vertical asymptotes at $x = -1$ and $x = 1$ (both approaching ∞), and approaches $-\infty$ on both "ends". There should be nothing graphed between $x = -1$ and $x = 1$, because the function is not defined there.

Section 9.3

1. Part (a): $3^{x-1} = 2 \implies \ln(3^{x-1}) = \ln 2 \implies (x-1)(\ln 3) = (\ln 2) \implies x - 1 = \frac{\ln 2}{\ln 3} \implies x = \frac{\ln 2}{\ln 3} + 1$.
Part (b): $3^{x-1} = 2 \implies \log_3(3^{x-1}) = \log_3 2 \implies x - 1 = \log_3 2 \implies x = (\log_3 2) + 1$.
The answers are the same by the base conversion formula.

Section 9.3 (continued)

2. Hint: To do part (a) you will need to use the fact that $\ln(4(2^x)) = (\ln 4) + \ln(2^x)$.
6. (a) 1; (b) ∞; (c) 0.
8. Hint: $\frac{d}{dx}(\ln|f(x)|) = \frac{f'(x)}{f(x)}$.
16. $\frac{3^x-1}{3^x+1} = 4 \implies 3^x - 1 = 4(3^x+1) \implies -3(3^x) = 5 \implies x = \log_3(-\frac{5}{3})$, which does not exist; therefore there are no solutions.
19. $\lim_{x\to 0}(x^{\ln x}) \to 0^{-\infty} \to \frac{1}{0^\infty} \to \frac{1}{0} \to \infty$.
23. This limit $\to 0^0$, so instead we'll look at:
$\lim_{x\to 2}(\ln((x-2)^{x^2-4})) = \lim_{x\to 2}((x^2-4)\ln(x-2))$ [[this $\to 0(-\infty)$]] =
$$\lim_{x\to 2}\frac{\ln(x-2)}{(\frac{1}{x^2-4})} \text{ [[this} \to \tfrac{0}{0}]] \overset{\text{L'H}}{=} \lim_{x\to 2}\frac{(\frac{1}{x-2})}{-(x^2-4)^{-2}(2x)} = \lim_{x\to 2}\frac{-(x^2-4)}{(x-2)(2x)} =$$
$$\lim_{x\to 2}\frac{-(x-2)(x+2)}{(x-2)(2x)} = \lim_{x\to 2}\frac{-(x+2)}{2x} = \frac{-4}{4} = -1.$$
Therefore the original limit is $\lim_{x\to 2}((x-2)^{x^2-4}) = e^{-1} = \frac{1}{e}$.
24. e.
25. 1.
27. $\lim_{x\to\infty}(\frac{1}{x+1})^x \to 0^\infty \to 0$.
28. This limit $\to 1^\infty$, which is indeterminate, so instead we'll look at:
$\lim_{x\to\infty}(\ln((\frac{x}{x-1})^x)) =$ [[rewrite, use L'H, then algebra]] $= \lim_{x\to\infty}\frac{x^2}{x^2-1} = 1$.
Thus the original limit is equal to $e^1 = e$.
35. Take the natural logarithm of both sides and use algebra to get:
$\ln|y| = 2x + 4\ln|x^3-2| - \ln|x| - \ln|3e^{5x}+1|$.
Then differentiate both sides to get: $\frac{1}{y}y' = 2 + \frac{4(3x^2)}{x^3-2} - \frac{1}{x} - \frac{15e^{5x}}{3e^{5x}+1}$.
Therefore: $y' = (\frac{e^{2x}(x^3-2)^4}{x(3e^{5x}+1)})(2 + \frac{4(3x^2)}{x^3-2} - \frac{1}{x} - \frac{15e^{5x}}{3e^{5x}+1})$.
37. $y' = \sqrt{\frac{x^2(3x+1)^{99}}{4x^2-3x+2}}(\frac{1}{2})(\frac{2}{x} + \frac{99(3)}{3x+1} - \frac{8x-3}{4x^2-3x+2})$.
40. Since there is a variable in both the base and the exponent we must use logarithmic differentiation. Let $y = x^{\ln x}$; then $\ln y = \ln(x^{\ln x}) = (\ln x)(\ln x) = (\ln x)^2$. Differentiating both sides we get $\frac{1}{y}y' = 2(\ln x)(\frac{1}{x})$, and thus $y' = x^{\ln x}(2)(\ln x)(\frac{1}{x})$.
43. Let $y = (\frac{x}{x-1})^x$. Then $\ln y = x\ln(\frac{x}{x-1}) = x(\ln x - \ln(x-1))$. Differentiating both sides gives $\frac{1}{y}y' = (1)(\ln x - \ln(x-1)) + x(\frac{1}{x} - \frac{1}{x-1})$. Therefore $y' = (\frac{x}{x-1})^x(\ln x - \ln(x-1) + 1 - \frac{x}{x-1})$.
47. **Proof:** Let $y = Ax^k$. We want to show that $y' = kAx^{k-1}$. Taking logs of both sides gives us $\ln|y| = \ln|Ax^k| = \ln|A| + k\ln|x|$. Differentiating both sides yields $\frac{1}{y}y' = \frac{k}{x}$, so $y' = y(\frac{k}{x}) = (Ax^k)(\frac{k}{x}) = kAx^{k-1}$. ■
51. **Proof:** Let $y' = \frac{f(x)}{g(x)}$. Taking logs of both sides gives $\ln|y| = \ln|f(x)| - \ln|g(x)|$; differentiating gives $\frac{1}{y}y' = \frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}$, so $y' = \frac{f(x)}{g(x)}(\frac{f'(x)}{f(x)} - \frac{g'(x)}{g(x)}) = \frac{f(x)}{g(x)}(\frac{f'(x)g(x)-g'(x)f(x)}{f(x)g(x)}) = \frac{f'(x)g(x)-g'(x)f(x)}{(g(x))^2}$. ■